

Three Infinities in Early Modern Philosophy

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Many historical and philosophical studies treat infinity as an exclusively quantitative notion, whose proper domain of application is mathematics and physics. The main aim of this paper is to disentangle, by critically examining, three notions of infinity in the early modern period, and to argue that one—but only one—of them is quantitative. One of these non-quantitative notions concerns being or reality, while the other concerns a particular iterative property of an aggregate. These three notions will emerge through examination of three central figures in the period: Locke (for quantitative infinity), Descartes (ontic infinity), and Leibniz (iterative infinity).

1. Introduction

It is widely recognized that Aristotle's discussion of infinity, in book VI of the *Physics*, served as the starting point and benchmark for philosophical work on infinity for centuries to come. Although several aspects of Aristotle's treatment had fallen out of favour by the early modern period—in particular, his strictures against 'actual infinity'—one central tenet is commonly thought to have remained firmly in place, namely, that the proper domain of application of infinity is that of quantity, and that the study of infinity belongs to the study of quantities, viz, mathematics and physics.¹ In this Aristotelian spirit, many historical and philosophical studies treat infinity as an

¹ 'The science of nature is concerned with spatial magnitudes and motion and time, and each of these at least is necessarily infinite or finite, even if some things dealt with by the science are not, for example, a quality or a point...Hence it is incumbent on the person who specializes in physics to discuss the infinite and to inquire whether there is such a thing or not, and, if there is, what it is' (*Physics* 202b30-35).

exclusively quantitative notion.² And even when a non-quantitative, alternative notion is acknowledged—for example, by making a distinction between a ‘mathematical’ infinity on the one hand and a ‘metaphysical’ infinity on the other³—it is not always made clear what the latter amounts to, and in what sense it is non-quantitative. In effect, the Aristotelian doctrine is not so much challenged as it is reinforced.

A main aim of this paper is to disentangle, by critically examining, three notions of infinity found in the seventeenth century, and to argue that one—but only one—of them is quantitative. A second main aim is to clarify what it is for a notion of infinity to be quantitative—that is, to specify conditions under which a notion of infinity qualifies as quantitative—and, hence, to clarify what it is for a notion of infinity to be *non*-quantitative.

Though each of the three notions I will discuss concerns unlimiteness, and in this sense is a notion of infinity, they are nonetheless distinct: while one concerns the size or measure of a given quantity—and in this sense is quantitative—the other two do not. One of the non-quantitative notions concerns being (ontic infinity), while the other concerns a particular iterative property of an aggregate (iterative infinity).

These three notions will emerge through examination of three central figures in the early modern period, each of whose work brings to the fore in a particularly clear and illuminating way one of our three notions: Locke (for quantitative infinity), Descartes (for ontic infinity), and Leibniz (for iterative infinity). My intention is not to suggest that they are the only figures in the period whose treatment of infinity merits close examination.⁴ Rather, by focusing on these three figures

² See, for example, Koyré (1957), Benardete (1964), Kretzmann (1982), Duhem (1987), and Mancosu (1996). Prominent efforts to evade this tendency can be found in post-Kantian German and existentialist philosophy (for example, Levinas 1969); *cp.* Franks (2006).

³ Moore (1990) invokes the terms ‘mathematical’ and ‘metaphysical’ to delineate two clusters of concepts that have dominated historical discussions of infinity: *boundless, endless, unlimited, and immeasurable*, on the one hand; *complete, whole, unity, universal, absolute, perfect, self-sufficient, and autonomous*, on the other hand. Moore’s aim is not to proffer an analysis or account of these concepts, but rather to point to a family resemblance among the concepts in each cluster.

⁴ Other important treatments of infinity in the period include Galileo’s, Spinoza’s, Conway’s, and Newton’s, to mention just a few. While I briefly touch on some of them below (see, for example, the discussion of Newton in note 14 and §2.3, and of Spinoza in note 34), it is impossible to do full justice to them in the compass of a single article. For fuller discussion, see the books by Moore (1990) and Mancosu (1996, especially ch. 5).

and setting them side by side, I aim to gain the perspective necessary to identify and clarify the three notions just mentioned.

Distinguishing these notions is significant for early modern scholarship. For it allows us to recognize that scholarly work on infinity in the period is sometimes either underspecified (when scholars do not make clear what notion of infinity they interpret a text as discussing) or mistaken (when they interpret a text as discussing one notion although it is discussing another).⁵ It may be of interest also for contemporary thought, insofar as reflection on the varieties of infinity in seventeenth-century philosophy may bring to light neglected yet possibly attractive resources for theorizing about infinity.

In §2, I articulate the rudiments of Locke's account of infinity and identify two conditions on infinite quantity. In §3, I formulate Descartes' notion of *ontic* infinity, while §4 explains Leibniz's *iterative* conception of infinity, which is subsequently contrasted with both ontic infinity and, in §5, with quantitative infinity.

2. Locke and quantitative infinity

At the opening of his discussion of infinity in the *Essay Concerning Human Understanding*, Locke explicitly locates infinity within the domain of quantity and the quantitative. In what I will refer to as *the quantity passage*, Locke writes:

Finite, and *Infinite*, seem to me to be looked upon by the Mind, as the *Modes of Quantity*, and to be attributed primarily in their first designation only to those things, which have parts, and are capable of increase and diminution, by the addition or subtraction of any the least part: and such are the *Ideas* of Space, Duration, and Number.⁶ (*Essay* II.xvii.1)

Locke's emphasis on quantity has not gone unnoticed by scholars, who view it as vital to Locke's empiricist (and anti-Cartesian) project of locating the origins of all our ideas in sensation and reflection. Through the identification of quantity as the paradigm instance of infinity—something that, unlike God (Descartes' paradigm, as explained

⁵ Several examples are identified in footnotes below.

⁶ Citations from Locke are from *An Essay Concerning Human Understanding* (abbreviated as 'Essay'), given by book, chapter, and paragraph number; or from drafts for the *Essay*, cited by draft (for example, 'Draft A') and section number. Full bibliographical information is given in the bibliography.

below, in §3), is given to us via sensation and reflection—the path to an empiricist account of the origin of our idea of infinity is cleared.⁷

Most scholars have focused almost exclusively on how, and with what degree of success, Locke accounts for this origin, declining to offer an interpretation of that which so originates. That is, they do not attempt to say what Locke's quantitative notion of infinity is, perhaps taking it to be obvious.⁸ I propose to address this latter—and, in a sense, more basic, though by no means obvious—topic. Thus the question I wish to consider is not 'Where does our idea of infinite quantity come from?', but rather 'Under what conditions is a quantity infinite?'

I will pursue an answer not by looking for a strict definition of infinity, or of quantity, but rather by seeking to understand how infinite quantity behaves or functions.⁹ I will begin with a clarification of Locke's notion of quantity, which relates it to measure, and in turn to number (§2.1). Subsequently, I will identify two conditions the satisfaction of which is (separately) necessary and sufficient for a quantity to be infinite, according to Locke (§2.2); the behaviour of infinite quantities is determined by these two conditions. Finally, I will briefly consider a tempting but ultimately untenable alternative interpretation of Locke's position (§2.3).

2.1 *Quantity, measure, and number*

Locke does not define quantity, in the *Essay* or elsewhere. But he does provide a mark that distinguishes quantities such as duration and space from qualities such as colour and heat: whereas quantities allow for 'exact measures', qualities do not.¹⁰ When comparing two shades of white, for example, it is not always possible to determine whether they are equal or unequal, and if unequal, by how much. In

⁷ See, for example, Aaron (1955, p. 167): '[The discussion of infinity in the *Essay*] is solely an attempt to demonstrate that the concept of infinity contains in it nothing not ultimately derived from sensation and reflection. To prove this Locke first endeavours to show that the only conception of infinity which can seriously be considered by us is the quantitative'. *Cp.* Adams (1975, pp. 81-2), Rogers (1995, p. 64), Rickless (2014, p. 62), Downing (2015), and citations in note 32.

⁸ In §2.3 I articulate and subsequently reject an interpretation of Locke's notion of infinity suggested by some of the scholarly work on the topic.

⁹ I borrow the expression from Dawson (1959, p. 303).

¹⁰ The expression 'exact measure' is from *Essay* IV.ii.11. In the same vein, Locke there writes: 'We have not so nice and accurate a distinction of their [qualities'] differences as to perceive, or find ways to measure, their just equality, or the least differences' (*ibid.*; *cp.* Draft A, §11).

the case of quantities, on the other hand, the equality or inequality of two instances can be measured.¹¹ In short, quantities, but not qualities, are measurable.

As with quantity, Locke does not define measure. Yet he has a longstanding interest in the subject,¹² and some of his commitments can be gleaned from his discussions of measure in the realm of number, duration, and space. Hence Locke writes (in what I will refer to as *the measure passage*):

This further is observable in number, that it is that which the mind makes use of in measuring all things that by us are measurable, which principally are expansion and duration; and our idea of infinity, even when applied to those, seems to be nothing but the infinity of number. (*Essay* II.xvi.8)

Locke here makes two interrelated—and theoretically significant—points about measure. First, number is the yardstick for measure: a quantity's measure is specified by means of number. Second, the same notion of measure, the yardstick for which is number, applies both in the infinite and in the finite case; hence the measure of *any* quantity, whether finite or infinite, can be specified by means of numbers. I believe that these two points serve as the bedrock of Locke's quantitative notion of infinity. Let us consider each in turn.

The first point is elucidated elsewhere, in a discussion of duration and its measure. After making two preliminary points, about how the idea of duration is formed,¹³ Locke writes (in what I will refer to as *the duration passage*):

Thirdly, by sensation observing certain appearances, at certain regular and seeming equidistant periods, we get the ideas of certain lengths or measures of duration, as minutes, hours, days, years, and so on. Fourthly, by being able to repeat those measures of time, or ideas of stated length of duration, in our minds, as often as we will,

¹¹ See *Essay* IV.ii.10; *cp.* *Essay* II.xvi.4. Locke suggests that we *could* measure the precise degree of a quality, such as a particular shade of white, if we could only observe the size, figure, number, and motion—that is, the quantitative properties—of the particles or corpuscles that serve as its causal basis (*Essay* IV.ii.11). This seems in line with the claim that quantities but not qualities allow for exact measure, for arguably what would be measurable in this case is not the quality itself, but its underlying, quantitative, causal basis.

¹² For an extensive discussion of Locke's interest in measurement, see Anstey (2016).

¹³ The first is that we acquire the idea of duration by reflection on the succession of our ideas; the second is that this succession has a measurable length. *Cp.* *Essay* II.xiv.2-4 and Yaffe (2011).

we can come to imagine duration, where nothing does really endure or exist; and thus we imagine to-morrow, next year, or seven years hence.¹⁴ (*Essay* II.xiv.31)

Locke's primary concern here is to refute the claim that duration requires or presupposes something that endures in it. To that end, he is inviting his reader to perform a thought experiment, by imagining 'empty' duration, beyond anything enduring. The important point for our purpose is that this thought experiment involves imagining duration *with a certain measure*: 'tomorrow, next year, or seven years hence'.

What, according to Locke, is it to have a measure? We secure an answer to this question by considering how measure is determined. Determining the measure of duration—measuring it—is a two-step process. The first step is to specify a unit of measurement, such as a minute, an hour, a day, or a year. The second step is to determine how many iterations of this unit are needed to match the duration in question; or, equivalently, how many parts matching the unit the interval contains. The measure of the duration—that in which its measure consists—is the number of units this process yields. Hence the measure of the duration in which the earth completes one revolution around the sun (to use a familiar example) is 525,600 minutes; 8,760 hours; 365 days; or one year. A similar process (with the appropriate choice of unit) serves to determine the measure of spatial quantities such as length, area, and volume.¹⁵

This reveals an important feature of Locke's notion of quantity. We have seen that something is a quantity just in case it has a measure. Given Locke's approach to measure, this in turn is equivalent to the

¹⁴ Locke here collapses the traditional Aristotelian distinction between duration and time, treating the two as interchangeable. In that tradition, time is viewed as the measure (or as Aristotle puts it in *Physics* 219b1, the 'number') of duration; and, moreover, time, qua measure, is viewed as mind-dependent, whereas duration is not (*Physics* 223a22). Among early modern figures, Descartes and Spinoza both subscribe to this tradition. (See, for example, *Principles* 1.56 (AT VIII.A.26/CSM 1.211) and Spinoza's letter to Lodewijk Meyer from April 20th 1663 (Spinoza 2002, pp. 787-91).) However, Locke does not. In collapsing the distinction, Locke is arguably influenced by Newton, whose conception of time as 'absolute' does not square with a distinction between time and duration—and particularly, with time being mind-dependent. Newton writes: 'Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external and by another name is called duration' (Newton 1999, p. 408). It remains a disputed question whether Locke, like Newton, endorses absolutism regarding time and space—and if so, whether this position is compatible with his empiricism. I return to the ramifications of Newton's influence on Locke in §2.3.

¹⁵ As discussed in, for example, *Essay* II.xiii.2-4, *Essay* II.xiv.24, and *Essay* II.xvii.3.

claim that *something is a quantity just in case it is divisible into unit-length parts that can be numbered or counted*.¹⁶

2.2 Two conditions for infinite measure

With this clarification of Locke's notion of quantity, and its relation to measure and number, we are now in a position to appreciate the second point in the measure passage, namely, that number is the yardstick for measure in the infinite case as well as in the finite case. Doing so will put us in a position to formulate two conditions for when a quantity has infinite measure.

In a sentence immediately following the duration passage, Locke explicitly mentions the role of number in determining infinite duration:

[B]y being able to repeat ideas of any length of time, as of a minute, a year, or an age, as often as we will in our own thoughts, and adding them one to another, without ever coming to the end of such addition, any nearer than we can to the end of number, to which we can always add; we come by the idea of eternity. (*Essay* II.xiv.31)

As this indicates, the measure or size of eternity—infinite duration—is given by the same two-step process described above, by (first) specifying a unit of measurement and (second) numbering or counting the unit-parts of the quantity in question. Whereas a quantity is finite just in case it has *finitely* many unit-parts, a quantity is infinite just in case it has *infinitely many* unit-parts.¹⁷

¹⁶ The preceding remarks do not purport to deliver a full account of Locke's metaphysics and epistemology of measure. For example, they do not settle the metaphysical question of whether measure is what Locke calls a 'primary' or rather a 'secondary' quality. Similarly, they do not address the epistemological problem of ascertaining that a chosen unit of measure is constant and unchanging through the process of measuring, given that the only way to ascertain this would presumably be to measure it—thereby engendering regress. (For Locke's concern with this problem, see, for example, Draft B §41-42 and *Essay* II.xiv.18, as discussed by Anstey (2016).) Finally, they do not address the interesting question of whether and how this account can be extended to accommodate measures that involve non-natural (that is, rational or irrational) numbers.

¹⁷ Thus Locke accepts the possibility (though not necessarily the actuality; see §2.3) of quantities that are infinite in multitude as well as infinite in magnitude. A quantity is infinite in multitude if it has infinitely many parts, whereas it is infinite in magnitude if it has infinitely many parts and its measure is infinite. Hence a *finitely long* line segment is infinite in multitude but not in magnitude, whereas an *infinitely long* line segment is infinite in both multitude and magnitude. Locke's acceptance of the infinite in magnitude is not entirely standard in the period. Galileo, for example, arguably rejects it (as discussed in Levey 2015).

Of course, one cannot count infinitely many items in the way one standardly counts finitely many, namely, by *enumeration*, starting with 1 and continuing along the number line until running out of items to count—whereby the last number cited is the number of items (that is, how many there are). This method is inapplicable to the infinite case because, as Locke is aware, infinity is not itself a number on the number line.¹⁸ There is, however, another method of counting that is applicable to the infinite case.

This second method, which I will call *tallying*, is underwritten by the notion of equinumerosity. The intuitive idea is that if items in one collection can be paired with items in another collection without remainder, it can be concluded that there are *as many* of the former as of the latter; the two are equinumerous. Call the principle expressed by this idea the *Pairing Principle*, which can be formulated as follows:

Pairing Principle: There are as many *F*s as *G*s if and only if there is a pairing of *F*s with *G*s without remainder.

Tallying and its underlying Pairing Principle have a long, if partly unwritten, history, evident in the use of objects (tokens, notches, abaci, and so on) for counting across various cultures and epochs.¹⁹ Importantly, they also have a long history as implicit tools applied in mathematical practice, some of whose applications involve *infinite* collections of items matched with each other.²⁰ In other words, the Pairing Principle was often not restricted to what has finitely many unit-parts but was taken to apply to cases involving infinitely many unit-parts as well.

Tallying is clearly at play in Locke's discussion of eternity. When we consider infinite duration, Locke tells us, we recognize that its unit-parts mirror the natural numbers: just as there are more and more unit-parts of duration, so there are more and more numbers.²¹ Implied here is the thought that the two groups can be paired without

Leibniz's attitude towards the infinite in magnitude is discussed in an exchange between Carlin (1997), Brown (1998), Arthur (1999), Brown (2000), and Arthur (2001).

¹⁸ For Locke's explicit denial that infinity is a number, see *Essay* II.xvii.13. I return to this point below, in §5.

¹⁹ See Roche (1998, ch. 1).

²⁰ For discussion of such historical applications in Cavalieri and Galileo, to name two important examples from the early modern period, see Andersen (1985), Knobloch (1999), Mancosu (2009), and Levey (2015).

²¹ *Cp.* Dawson (1959, p. 306).

remainder. Consequently, Locke concludes, there are as many such parts in eternity as there are natural numbers, which is to say infinitely many.²²

Of course, this assumes that there are infinitely many natural numbers. But this claim is fairly secure, and below I will discuss Locke's own reason for endorsing it. In the meantime, let us formulate a first condition for when, according to Locke, a given quantity is infinite:

First Quantitative Condition: A quantity is infinite just in case there is a pairing without remainder between its unit-parts and the natural numbers.

Although Locke never articulates this condition as such, I believe that it follows naturally from the conjunction of four Lockean theses that emerged over the course of the foregoing discussion: first, infinity is a mode of quantity; second, there are direct links between quantity, measure, and measuring; third, there is continuity between measuring finite and infinite quantities; and fourth, in both cases number is that by which they are measured.

We have considered the possibility of tallying infinite quantities: pair their unit-parts with the natural numbers without remainder. At several points Locke suggests another method of measuring infinite quantities, which does not require a comparison with the natural numbers (or any other quantity that is independently known to be infinite), but with finite quantities—in particular, the quantity's own finite parts. This method of measuring does not to my knowledge have a name, but it is related to the longstanding principle known as *Euclid's Axiom* (because of its inclusion as a 'common notion' in Euclid's *Elements*):

Euclid's Axiom: A whole is greater than its proper parts.

This principle entails a second criterion for when a given quantity is infinite. For, intuitively, if a quantity has *arbitrarily large* finite parts, then it cannot be finite: it is infinite, having infinitely many unit-parts.²³

²² Similarly, tallying seems to be at work when Locke says: 'By being able to repeat the idea of any length of duration we have in our minds, with all the endless addition of number, we come by the idea of eternity. For we find in ourselves, we can no more come to an end of such repeated ideas than we can come to the end of number; which every one perceives he cannot' (*Essay* II.xvii.5). Cp. *Essay* II.xiv.26, II.xiv.30, II.xvi.8, and II.xvii.11. A parallel case concerning the infinity of space is made in *Essay* II.xiii.4, II.xv.2, and II.xvii.11.

²³ The scope of Euclid's Axiom is nowadays commonly qualified, so that it does not apply to infinite collections or sets (as witnessed by Dedekind's definition of an infinite set as

To illustrate, consider Locke's discussion of the infinity of the natural numbers:

For such an inexhaustible stock, number (of all other our ideas [*sic*]) most clearly furnishes us with, as is obvious to every one. For let a man collect into one sum as great a number as he pleases, this multitude, how great soever, lessens not one jot the power of adding to it, or brings him any nearer the end of the inexhaustible stock of number; where still there remains as much to be added, as if none were taken out. (*Essay* II.xvi.8)

In the first part of the second sentence of this passage, Locke highlights the fact that for any finitely large multitude of numbers one can think of, there is a greater one; since the entire multitude or 'stock' of numbers includes all such finite yet arbitrarily large multitudes as parts, it is therefore infinite. Different passages highlight this characteristic with regard to other infinite quantities. With regard to eternity, for example, Locke writes:

I can add one minute more till I come to sixty; and by the same way of adding minutes, hours, or years (that is, such or such parts of the sun's revolutions, or any other period whereof I have the idea) proceed in infinitum, and suppose a duration exceeding as many such periods as I can reckon, let me add whilst I will, which I think is the notion we have of eternity.²⁴ (*Essay* II.xiv.30)

For any finite interval of duration, there is a greater one; a duration that has all such arbitrarily large finite durations as parts ('a duration exceeding as many such periods as I can reckon...') is infinite duration, or eternity.

This enables us to formulate a second condition for when, according to Locke, a given quantity is infinite:

Second Quantitative Condition: A quantity is infinite just in case it has arbitrarily large finite parts.

For, again, if a quantity has arbitrarily large finite parts, it must itself be greater than all these parts; which is to say, it must be infinite.

equinumerous with one of its proper parts). We will return to Euclid's Axiom below, in §4, and will see that already in the seventeenth century, some (notably, Leibniz) saw a need to qualify it in this manner. However, importantly, not everyone in the period saw this need, including, I suggest, Locke.

²⁴ See also *Essay* II.xiii.4, II.xv.2-3, II.xvii.3 and II.xvii.9.

As passages such as the above make clear, the second condition plays an important role in Locke's discussion. In fact, the intuitive idea underlying it is arguably presupposed by Locke's characterization of quantities (in the quantity passage and elsewhere) as things which 'have parts, and are capable of increase and diminution, by the addition or subtraction of any the least part'. Since a quantity is greater than any of its parts, it will be made smaller by reducing it to what was formerly one of its parts; and it will be made larger by expanding it to include as a part what *it* formerly was.

If this is correct, then we have explained in what sense, and why, Locke views infinity as a mode of quantity: a quantity is infinite in the sense that it has infinite measure. This is the case, moreover, when a quantity satisfies two distinct conditions. Although each (separately) specifies what is necessary *and* sufficient for being an infinite quantity, the two conditions emphasize different features of how infinite quantity behaves: the first concerns the equality between it and other infinite quantities, and the second the inequality between it and its finite parts.

To be clear, to simply identify these conditions is not yet to argue that anything satisfies them. That is, to argue that Locke has a notion of infinite quantity is not yet to argue that he believes that infinite quantities in fact exist. As we will see in a moment, it is a matter of some controversy whether Locke thinks that space and time, to mention two of the most natural candidates, are in fact infinite.²⁵ But it is to argue that something is, or could be, an infinite quantity so long as, and only so long as, it meets these two conditions.²⁶

²⁵ Other natural candidates are the divine attributes, such as omniscience and omnipotence. In §3, we will confirm that Locke thinks that such attributes are infinite in the quantitative sense captured by our two conditions.

²⁶ It might be argued that this is denied when Locke says that 'we cause great confusion in our thoughts, when we join infinity to any supposed idea of quantity the mind can be thought to have, and so discourse or reason about an infinite quantity, as an infinite space, or an infinite duration' (*Essay* II.xvii.7). As I read this comment, however, Locke is not issuing a blanket warning against the possibility of there being an infinite quantity, but against the possibility of there being an idea of such quantity, when that idea is understood as the product of the mind's compounding or composition of its simple ideas, by analogy to the way the mind produces ideas of finite quantities. (See, for example, *Essay* II.xvi.2.) This is untenable because the mind does not have the resources to compound infinitely many simple ideas; the passage just cited continues: '[T]o have actually in the mind the idea of a space infinite, is to suppose the mind already passed over, and actually to have a view of all those repeated ideas of space which an endless repetition can never totally represent to it; which carries in it a plain contradiction' (*Essay* II.xvii.7). How Locke thinks the idea of infinity *is* to be understood is a matter of controversy, to which I return below, in note 32.

It is important to emphasize that the resulting Lockean notion of infinity is not ad hoc or arbitrary. After all, both conditions are underwritten by longstanding and intuitive principles about how quantities behave in the finite case, which Locke extends to the infinite case based on general considerations involving measure and measuring. In effect, by holding that infinite quantities are governed by the two conditions we have identified, Locke holds that they behave like finite quantities. Below we will attempt to articulate this point in a more formal key. But I submit that even in the absence of such formal articulation, what has been said to this point is sufficient to fulfil our initial goal of explaining Locke's quantitative notion of infinity, and to do so in a way that, as we shall see, provides the resources needed to state what it is for a notion of infinity to be *non*-quantitative.

2.3 *An alternative interpretation*

Before turning to this project, I would like to conclude this section by briefly considering a natural, but to my mind ultimately untenable, alternative interpretation of Locke's treatment of infinity. This interpretation endorses two main theses: first, infinity is an iterative property that finite quantities have, collectively, when for each one of them, a greater finite quantity exists; second, there is and could be no single quantity that is greater than all finite quantities (as an infinite quantity, as described above, would be). Together, these two claims imply that nothing satisfies, or could satisfy, the two conditions we have identified: nothing can have as many unit-parts as the natural numbers, and nothing can have arbitrary large finite parts.

While this iterative interpretation, as I will call it, has not (to my knowledge) been explicitly formulated in the literature, it is arguably implicit in many scholarly discussions of Locke's treatment of infinity.²⁷ Its proponents could point to some passages that seem to support it, but such textual evidence is inconclusive at best,²⁸ and there

²⁷ For example, Dawson (1959, p. 305) seems to allude to it when he writes: 'The essential point in considering infinity as a process of number generation is that the process is non-ending... That is, for his account of infinity Locke requires, not only the method of constructing numbers indicated in Chapter xvi, but also the postulate "for every number there is a greater number".'

²⁸ For example: 'All the ideas that are considered as having parts, and are capable of increase by the addition of any equal or less parts, afford us, by their repetition, the idea of infinity; because, with this endless repetition, there is continued an enlargement of which there can be no end' (*Essay* II.xvii.6). Notice that this passage is in fact neutral between the two interpretations: on both of them it is true that for any finite quantity, there is or can be a greater one. The difference is that the interpretation proposed here asserts, whereas the

are several passages that appear to me to lend unequivocal support to the quantitative interpretation I have been developing.²⁹ Rather than debate these passages, however, I wish to focus on what is, to my mind, the central objection to the iterative interpretation, namely, that it cannot accommodate Locke's commitments regarding the ontological status of space and time.

It is generally agreed among scholars that whereas in early drafts of the *Essay* Locke preferred a relationist view of space and time, on which they are not entities in their own right but mere relations between spatially- and temporally-located entities, by the time of the publication of the *Essay* he had come to accept, or at least to accept as possible, an absolutist position on which space and time are real, singular, infinite entities.³⁰ However, if the iterative interpretation were correct, Locke would have had to rule out Newtonian absolutism simply on conceptual grounds; for on that interpretation, there simply can be no real, singular entity that is infinite. Locke's acceptance of the possibility, if not the truth, of absolutism is therefore strong (perhaps decisive) evidence against the iterative interpretation.³¹

The quantitative interpretation proposed here, on the other hand, has no problem accommodating Locke's commitments regarding the ontology of space and time. For it is precisely the possibility of a real, singular entity that is infinite that this interpretation captures.

As should be clear, my aim has not been to provide a comprehensive analysis of Locke's notion of infinity (and, in particular, our idea thereof³²) but to clarify the sense in which infinity is quantitative,

alternative interpretation denies, that there can also be an infinite quantity. The passage affirms what is not in dispute, and says nothing about what is in dispute.

²⁹ Recall that Locke not only says that for any natural number there is a greater one, but also that there is an 'inexhaustible stock' of numbers to which they all belong (*Essay* II.xvi.8, cited above). Likewise, he not only says that for any finite duration a greater duration exists, but also speaks of a duration that exceeds all such finite durations—to wit, infinite duration or eternity (*Essay* II.xiv.30, also cited above). And elsewhere, Locke speaks of space and duration themselves as 'boundless oceans' and 'infinite abysses' (*Essay* II.xiv.5-6).

³⁰ This change is often attributed to Locke's encounter with Newton's *Principia*, which he read shortly after its publication in 1687 (Locke's *Essay* was published in 1690), and in which Newton defends an absolutist position. See Thomas (2016) for a helpful overview of the debate regarding Locke's alleged absolutism. See Gorham and Slowik (2014) for the implications of Locke's Newtonianism for his empiricism. Finally, see Downing (1997) for discussion of the broader extent of Newton's influence on Locke.

³¹ We will return to the iterative view below, in §5, when discussing Leibniz.

³² Locke's treatment of this idea is notoriously fraught with difficulties (for example, his characterization of it as somehow 'negative', or his claim that it is 'endlessly growing'). For

according to Locke. Insofar as the quantitative notion of infinity is vital to Locke's empiricist, anti-Cartesian project in the *Essay* (as explained above), such clarification furthers our understanding and appreciation of this project. Moreover, as I hope to show next, it puts us in a position to contemplate other, *non*-quantitative notions of infinity in contrast.

3. Descartes and ontic infinity

It was noted earlier that whereas for Locke quantity is the paradigmatic and exclusive province of infinity, for Descartes God is.³³ Locke does not deny that divine attributes are infinite, but simply treats them as instances of infinite quantity. After claiming (in the quantity passage) that infinity applies in its 'first designation' to quantities, such as number, duration, and space, Locke goes on to say that the self-same notion applies to God as well:

[W]hen we apply to that first and supreme Being our idea of infinite...we have no other idea of this infinity but what carries with it some reflection on, and imitation of, that number or extent of the acts or objects of God's power, wisdom, and goodness... (*Essay* II.xvii.1)

This point is illustrated in another passage, in which Locke explains the infinity of God's knowledge, viz, his omniscience:

If I find that I know some few things, and some of them, or all, perhaps imperfectly, I can frame an idea of knowing twice as many; which I can double again, as often as I can add to number; and thus enlarge my idea of knowledge, by extending its comprehension to all things existing, or possible...to that vastness to which infinity can extend them. (*Essay* II.xxiii.34)

Some differences notwithstanding, Locke is employing here the same two-step process for measuring knowledge as he employs for measuring duration, by first, specifying a unit of measurement—in this case, an item of knowledge—and second, determining how many iterations

further discussion, see Aaron and Walters (1965), Ott (1997), Szabó (2000), Anstey (2011, ch. 6), and Rickless (2014, §4.5).

³³ References to Descartes' works cite the volume and page number in Descartes (1996) (abbreviated 'AT'), followed by the volume and page number in Descartes (1985–1992), vols. 1 and 2 (abbreviated 'CSM'), or by the page number in vol. 3 (abbreviated 'CSMK'). I use the following abbreviations for specific works by Descartes: '*Meditations*' for *Meditations on First Philosophy*, and '*Principles*' for *Principles of Philosophy*.

of this unit constitute God's knowledge. God's knowledge is infinite insofar as it meets the same conditions as other infinite quantities meet: it includes as many unit-parts as there are numbers ('I can double again, as often as I can add to number'), and those parts are arbitrarily large ('extending its comprehension to all things existing, or possible'). The same is true for the non-epistemic dimensions of God's infinity, such as omnipotence and benevolence. In this way, for Locke, God's infinity is simply an instance of quantitative infinity.

Locke's approach here stands in stark contrast to Descartes' stated position in the *Principles of Philosophy* that he 'reserve[s] the term "infinite" for God alone' (*Principles* 1.27; AT 8A.15/CSM 1.201).³⁴ In saying this, Descartes does not mean to deny that other entities may be non-finite or unlimited; however, in such cases he uses the term 'indefinite'. Some examples of indefinite entities are listed in the following passage:

There is, for example, no imaginable extension which is so great that we cannot understand the possibility of an even greater one; and so we shall describe the size of possible things as indefinite. Again, however many parts a body is divided into, each of the parts can still be indefinitely divisible. Or again, no matter how great we imagine the number of stars to be, we still think that God could have created even more; and so we will suppose the number of stars to be indefinite. (*Principles* 1.26, AT 8A.15/CSM 1.201)

It is clear from this list that the term 'indefinite' applies to quantities, such as space and number, or things that can be made greater by the addition of parts.³⁵ The term 'infinite', on the other hand, which

³⁴ Locke's approach also stands in stark contrast to Spinoza's, who, like Descartes, holds that there is a type of infinity that applies to God alone; it is, therefore, distinct from the type of infinity that applies to quantities. (See Spinoza's letter to Lodewijk Meyer from April 20th 1663 (Spinoza 2002, pp. 787-91).) As I lack the space to discuss both philosophers here, I have chosen to focus on Descartes, since whereas recent studies have suggested that Spinoza's treatment of infinity is non-quantitative (see in particular Nachtomly 2011; and contrast Gueroult 1968, p. 404), scholars tend to view Descartes' notion through a purely quantitative lens. (Two examples of this tendency are cited in note 42.) One of my aims here is to offer an alternative interpretation of Descartes' treatment that has not received the attention that I believe it deserves.

³⁵ I examine Descartes' notion of the indefinite in detail in Schechtman (2018), so I will not discuss it further here, beyond emphasizing the following two points. First, the indefinite is a type of unlimitedness that applies to quantities, such as number; second, it allows Descartes to perform mathematical operations on non-finite quantities—as in, for example, Descartes' demonstration, in the course of discussing Zeno's paradoxes, that the sum of the series $1/10 + 1/100 + 1/1000 + \dots = 1/9$ (letter to Clerselier from June/July 1646, AT IV 445-6/CSMK 290-1; cf. Leibniz's treatment of the sum of an infinite series, discussed in note 54 below). As this

Descartes reserves for God alone, can plausibly be expected to denote a *non*-quantitative type of unlimitedness.

This expectation is corroborated by the close connection Descartes draws between infinity, on the one hand, and being or reality, on the other. In particular, in his letter to Clerselier from 23 April 1649, Descartes explains this equation and what he takes it to imply about our idea of infinity:

I say that the notion I have of the infinite is in me before that of the finite because, by the mere fact that I conceive being or that which is, without thinking whether it is finite or infinite, what I conceive is infinite being; but in order to conceive a finite being, I have to take away something from this general notion of being, which must accordingly be there first.³⁶ (AT 5.356/CSMK 377)

Descartes says here that he understands the idea of the infinite as the idea of being *simpliciter*—being in general, or what *is*, neither limited nor qualified—and, by contrast, the idea of the finite as the idea of what *is* qualified or limited. There are reasons to think that this position carves out a non-quantitative notion of the infinite—which, because of the connection to being, I will label *ontic infinity*.

To appreciate these reasons, it will be helpful to focus on Descartes' proclamation, central to the argument for God's existence in the Third Meditation, that being (or reality) comes in degrees: an infinite substance has 'more reality' than a finite substance, which in turn has 'more reality' than a mode. Initially, Descartes' willingness to speak of degrees of reality may suggest that being is a quantitative notion: that it is measurable, and that different entities can possess more or less—hence, a greater or lesser *measure*—of it. However, elsewhere Descartes explicates this manner of speaking in a way that indicates otherwise. For Descartes, the different degrees of reality possessed by infinite substance (most reality), finite substance (intermediate reality), and mode (lowest reality) are due to these entities' membership in different ontological categories. These categories differ insofar as they imply different dependence relations: modes depend on substances; finite substances do not depend on modes, but they do depend on the

makes clear, the claim that Descartes has a non-quantitative notion of infinity (or unlimitedness), which applies to God alone, does not imply that Descartes has no other notion of infinity that applies to quantities.

³⁶ See also the Third Meditation (AT 7.46/CSM 2.31).

one infinite substance (God), which is itself absolutely independent (that is, it depends on nothing else whatsoever).³⁷

In short, Descartes views infinity as the highest, or unqualified, degree of being, where the latter is equivalent to absolute independence. Finitude, on the other hand, is a lower, or qualified, degree of being, where this is understood as qualified or relative independence.³⁸

Category	Status	Degree of Being	Dependence Relations
Infinite substance	Infinite	Highest	Absolutely independent
Finite substance	Finite	Intermediate	Dependent on God; independent of modes
Mode	Finite	Lowest	Dependent on a finite substance

Further evidence for this proposal comes when Descartes speaks of the relation between the infinite and the finite. In the letter to Clerselier, quoted above, Descartes says that one has to ‘take something away’ from the idea of the infinite in order to think of the finite. Elsewhere, Descartes says: ‘[T]he limitation which makes the finite different from the infinite is non-being or the negation of being [*non ens, sive negatio entis*]’ (letter to Hyperaspistes from August 1641; AT 3.427/CSMK 192). By ‘negation’ or ‘limitation’, Descartes does not appear to have in mind subtraction (or diminution)—that is, the taking away of an aspect, piece, element, or part. Instead, what finitude lacks is the *independence* that infinity requires; consequently, limitation or negation is properly understood in terms of dependence.

While there is more to be said, I believe that we have enough of Descartes’ position in view to see that his notion of infinity is non-quantitative. For such infinity clearly does not obey the two conditions governing infinite measure, repeated here:

³⁷ See the Third Meditation (AT VII.40/CSM 2.28) and the Third Replies (AT 7.185/CSM 2.130). In the latter passage, Descartes alludes to the possibility of a fourth degree of reality, possessed by ‘real qualities or incomplete substances’, intermediate between the degrees possessed by modes and by finite substances (*ibid.*). And it might be argued that there are additional degrees of reality, corresponding to other ontological categories that Descartes acknowledges, such as attributes. (See O’Neill (1987, p. 234) for a suggestion along these lines.) While I will continue to speak of three degrees of reality, nothing in the argument depends on this exact number.

³⁸ I discuss Descartes’ treatment of the relevant dependence relations in Schechtman (2016).

First Quantitative Condition: A quantity is infinite just in case there is a pairing between its unit-parts and the natural numbers.

Second Quantitative Condition: A quantity is infinite just in case it has arbitrarily large finite parts.

The lower degrees of being that characterize finite beings (that is, finite substances and modes) are not *parts* of the highest degree of being that characterizes God. There is simply no credible sense of ‘part’ in which relative independence (that to which lower degrees of being are equivalent) can be said to be a *part* of absolute independence (that to which the highest degree of being is equivalent).

Now, even if we were to grant for the sake of argument that parthood holds in this case, perhaps in some attenuated sense,³⁹ I believe the conditions governing infinite quantities would still not be satisfied. For, on Descartes’ view, there are three degrees of being.⁴⁰ So even if lower degrees of being could be said to be parts of the highest degree of being, there would still not be arbitrarily large parts, nor as many parts as there are natural numbers. If this is correct, Descartes’ willingness to speak of degrees of being, properly understood, provides evidence not for but against the hypothesis that his notion of infinity is quantitative.⁴¹

I propose, instead, that Descartes be interpreted as committed to an *ontic* notion of infinity, where such a notion satisfies the following condition:

Ontic Condition: A being is infinite just in case it has the highest degree of reality, where x has the highest degree of reality just in case x is absolutely independent.

³⁹ For example, perhaps it could be coherently maintained that insofar as the collection of entities with respect to which a finite substance is independent (*viz.* all finite beings) is a part of the collection of entities with respect to which an infinite being is independent (*viz.* all finite beings *and* the infinite being, itself), there is an attenuated, or derived, notion of parthood that holds between dependence and independence after all. Of course, it would be difficult to find such a notion in Descartes’ text.

⁴⁰ But recall note 37.

⁴¹ Interestingly, Locke himself warns against thinking of anything that allows for degrees as fitting the quantitative mould, and hence as capable of being (quantitatively) infinite. A prime example is colour. When explaining why ‘nobody ever thinks of infinite sweetness, or infinite whiteness’, Locke writes: ‘For to the largest idea of extension or duration that I at present have, the addition of any the least part makes an increase; but to the perfectest idea I have of the whitest whiteness, if I add another of a less or equal whiteness, (and of a whiter than I have, I cannot add the idea), it makes no increase, and enlarges not my idea at all’ (*Essay* II.xvii.6).

I have argued that Descartes is committed to this condition, and that its satisfaction does not imply satisfaction of the two Lockean conditions for quantitative infinity. If correct, this indicates that Descartes accepts a non-quantitative, *ontic* notion of infinity.⁴²

This is, of course, simply an outline or sketch of Descartes' position; there is more that must be said to fully flesh out his ontic approach to infinity and its historical and philosophical underpinnings.⁴³ Yet enough has been said to establish my primary point, namely, that whereas the Lockean notion of infinity, including divine infinity, is quantitative, the Cartesian notion of infinity is not.

4. Leibniz and iterative infinity

Our discussion of Locke identified two necessary and sufficient conditions that govern infinite quantities, each of which is underwritten by an intuitive principle, the Pairing Principle in one case and Euclid's Axiom in the other. I claimed that the two are longstanding principles that were commonly used in mathematical practice. At the same time, it is well known that when the two principles are combined, they give rise to a puzzle or paradox about infinity—one that was very much alive in the early modern period. While the puzzle has ancient roots, it reappears in the seventeenth century as an argument in Galileo's 1638 *Discourses on the Two New Sciences*, and has since come to be known as 'Galileo's Paradox'. I believe that Descartes' notion of the indefinite anticipates a third notion of infinity in the period—distinct from both Locke's quantitative notion and Descartes' ontic notion—that arises from Leibniz's deep engagement with this paradox.

Galileo's Paradox can be formulated as the following argument:

⁴² Some scholars do interpret Descartes' notion of the infinite in quantitative terms. For example, Nelson and Nolan claim that 'the modern mathematical idea of the cardinality of the natural numbers functions in a way similar to the idea of complete infinity (God) in Descartes' philosophy' (2006, p. 108; *cp.* Beyssade 1992, p. 179). As the discussion in the main text makes clear, I think such interpretations are mistaken. Here I only have space to develop the ontic, non-quantitative interpretation that I believe is correct. I discuss alternative interpretations in detail in Schechtman (2018).

⁴³ For example, Descartes' position (on my interpretation) bears interesting affinities to a traditional view of God in medieval philosophy and theology. This view conceives God as identical to, or as possessing, 'Being' or 'being itself' [*ipsum esse*], and earthly creatures as possessing qualified, limited being derived from God, by whom they were created and on whom they depend. Hence, for example, Augustine proclaims: 'When I first came to know you, you raised me up so that I might see that what I was seeing is Being, and that I who was seeing it am not yet Being' (*Confessions* 7.10.16). *Cp.* Aquinas (*Summa Theologica* 1.4.2).

- (1) The squares of natural numbers are a proper part of the natural numbers; (Only some of the natural numbers are squares.)
- (2) *Euclid's Axiom*: A whole is greater than any of its proper parts;
- (3) There is a pairing of the squares of natural numbers with the natural numbers without remainder;
- (4) *Pairing Principle*: There are as many *F*s as *G*s if and only if there is a pairing of *F*s with *G*s without remainder;
- (5) Therefore, the natural numbers are both greater than and equal to the squares of natural numbers.

This paradoxical conclusion arises because, as is well known, Euclid's Axiom and the Pairing Principle yield incompatible verdicts when applied to infinite collections.

Nowadays the orthodox (if not sole) solution to Galileo's paradox is to hold that only the Pairing Principle—which is now often called the 'Bijection Principle'—applies to all quantities, infinite as well as finite, and to reject the applicability of Euclid's Axiom to infinite quantities. Although this response was already contemplated in the seventeenth century, it was not the orthodox approach at the time. In a much-quoted passage, Leibniz deliberately rejects it:

Hence it follows [from the paradox] either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity is itself nothing, that is, that it is not one [*unum*] and not a whole [*totum*].⁴⁴ (A 6.3.158)

Leibniz's own position, which he presents as the only alternative to holding that Euclid's Axiom is false, is that an infinite aggregate of elements is not a whole. If so, then infinite aggregates are not counterexamples to Euclid's Axiom, since the axiom simply does not apply to them. Moreover, it does not follow from premises 1 and 2 that there are more natural numbers than squares. For if the natural numbers, *qua* an infinite aggregate of elements, do not form a whole, then *ipso facto* they do not form a whole greater than one of its parts. In this

⁴⁴ References to Leibniz's work use the following abbreviations: 'GM' for Leibniz (1849-63), cited by volume and page; 'G' for Leibniz (1875-1890), cited by volume and page; 'A' for Leibniz (1923-80), cited by series, volume, and page; 'NE' for Leibniz (1983), cited by book, chapter, and section; and 'LoC' for Leibniz (2001).

way, Leibniz resolves Galileo's Paradox without abandoning Euclid's Axiom.

The ramifications of Leibniz's resolution of the paradox and of his position that infinite aggregates do not form wholes have been extensively explored, and I do not intend to discuss them all here.⁴⁵ Instead, I will focus on what his solution implies for his conception of infinity itself, and, in particular, how this conception relates to the quantitative and ontic notions of infinity identified above.

Leibniz's position that there are no infinite wholes implies that there is no infinite number either, since according to Leibniz, numbers are themselves wholes.⁴⁶ But if there is no infinite number, then there is no number of natural numbers, or of moments of time, or of parts of matter. Leibniz must provide a different way of accounting for their infinity than by saying that their number is infinite. He does so by holding that there is an infinity of things when 'there are always more than one can specify' (NE II.xvii.1). Elsewhere he elaborates:

When it is said that there are infinitely many terms [or things], it is not being said that there is some specific number of them, but that there are more than any specific number. (Letter to Bernoulli from 13 January 1699; GM 3.566)

In this passage, Leibniz distinguishes between two manners in which something can be said to be infinite, only one of which he accepts: in the first, something is infinite in case the number of its terms (or parts) is infinite; in the second, something is infinite in case for any number, it has more terms (or parts) than that number.

We encountered a similar distinction earlier, when discussing the iterative interpretation of Lockean infinity (in §2.3). That interpretation holds that Locke's notion of infinity consists in the (actual or possible) existence, for any finite quantity, of a greater quantity; but not in the (actual or possible) existence of a quantity that is greater than all finite quantities. I argued that this is not an adequate representation of Locke's view, given his commitment to, or open-mindedness about, the possibility of absolute space. We can now see that, by contrast, it identifies the crux of Leibniz's position (who, as a relationalist, firmly rejected absolute space).

⁴⁵ Cp. discussion of the paradox in Russell (1903), Levey (1998, 2008, and 2015), Brown (2005), Nachtomy (2011), van Atten (2011), Harmer (2014), and the citations in note 17.

⁴⁶ 'An integer number is a whole [*totum*] collected from unities' (cited in Grosholz and Yakira 1998, p. 99). See also A 6.3.463, A 6.3.477, A 6.3.503, and G 1.338.

At the same time, it should be clear that Leibniz here does not endorse Descartes' ontic condition, repeated below:

Ontic Condition: A being is infinite just in case it has the highest degree of reality, where x has the highest degree of reality just in case x is absolutely independent.

Elsewhere Leibniz, like Descartes, distinguishes between a notion of infinity that applies to quantities, such as number or duration, and one that applies to God.⁴⁷ Regarding the latter, he would arguably agree that it—divine infinity—consists in non-quantitative properties such as maximal reality and absolute independence. However, as should be clear from the focus on numbers in the passages we have considered, this is not the notion of infinity Leibniz is concerned with in the context of Galileo's Paradox.

If this is correct, we can reasonably conclude that Leibniz's notion of infinity is not the same as Descartes' ontic notion. I have claimed that it also differs from Locke's quantitative notion—or, more cautiously, I have provided *prima facie* reason to think it is different. In the next and final section, I will further develop my argument that Leibniz's and Locke's notions are distinct, and that the disagreement between the two philosophers arises from divergent attitudes towards infinite measure, and hence infinite quantity, in the Lockean sense investigated above.

5. Locke versus Leibniz on infinite measure

What does Leibniz affirm, and what does he deny, in passages on infinity such as the one just quoted? Richard Arthur, in his influential work on Leibniz's philosophy of mathematics, interprets Leibniz as denying the first and affirming the second of the following two formulas (where Fx stands for 'x is finite', and m and n range over natural numbers):

- (i) $\exists m \forall n (Fn \rightarrow m > n)$;
- (ii) $\forall n \exists m (Fn \rightarrow m > n)$.

Formula (i) states the existence of a number greater than any finite number—that is, the existence of an infinite number. Formula (ii) states that for any finite number, there is a number greater than *it*.

⁴⁷ As discussed in Nachtomy (2011) and Antognazza (2015).

Arthur claims—rightly, in my view—that while Leibniz rejects (i), he accepts (ii).⁴⁸ Again, as Leibniz tells us, to say that there are infinitely many terms is not to say ‘that there is some specific number...but that *there are more than any specific number*’.⁴⁹

While I think that Arthur’s interpretation is correct, I also believe that it is incomplete. We have seen that Locke, too, rejects the existence of an infinite number, *contra* (i).⁵⁰ Moreover, Locke accepts that for every finite number, a greater one exists, as asserted in (ii). If Leibniz’s position consisted only in rejecting (i) and accepting (ii), it would be indiscernible from Locke’s—which, *prima facie*, it is not. (Recall the end of the previous section.)

I propose that there is a third formula, which Locke accepts and Leibniz rejects. This formula states the existence of an infinite *measure*, in the Lockean sense discussed earlier. In order to identify this formula, let us first formally explicate the notion of measure operative in Locke’s account:

k is a measure of some quantity Q if and only if k is a mode of Q and:

- (1) $k = n$ for some natural number n , or
- (2) For every quantity P whose measure is some natural number n , $k > n$.⁵¹

⁴⁸ See, for example, Arthur (2001) and (2015).

⁴⁹ Elsewhere, Leibniz makes it clear that he endorses the *actual* rather than merely *potential* infinity of numbers. In the Aristotelian tradition, an entity is potentially infinite if it is actually finite, yet can be endlessly augmented; an entity is actually infinite if it is not actually finite. (See *Physics* 206a18-19.) Leibniz writes: ‘[B]odies are actually infinite, that is, more bodies can be found than there are unities in any given number’ (A 6.4.1393; cited in Antognazza 2015). Notice that Leibniz’s explication of the actual infinity of bodies, in the second part of the sentence, is an instance of formula (ii): for any specific number, there are more bodies than that number. Hence, formula (ii), and the iterative notion of infinity it captures, is tantamount to actual rather than merely potential infinity. See also Leibniz’s letter to Des Bosses from September 1706 (G 2.314-315) and discussion in Beeley (2008, pp. 199-200) and Antognazza (2015, pp. 7-9).

⁵⁰ See note 18 above.

⁵¹ As noted above, the measure of a given entity is always determined with respect to some unit. I assume here that a unit for measuring Q and P has been specified. I leave it open whether a quantity Q , of which measure is a mode, is itself a substance or a mode, since the quantity passage (from which the expression ‘mode of quantity’ is taken) seems to me ambiguous on this matter. I am grateful to Shyam Nair for pressing me on this point.

This explication captures two of the central points that emerged from our earlier discussion of Locke. First, number is the yardstick for measure in the infinite case as well as in the finite case. Here, measure is said to be *either* a natural number (in the finite case), *or* a mode of quantity that is greater than any given natural number. Second, a quantity is infinite just in case it satisfies the two quantitative conditions we have identified: it has as many unit-parts as there are natural numbers, and it has arbitrarily large finite parts. A quantity that satisfies these two conditions will also satisfy (2) in the above explication: it will have as many unit-parts as there are numbers (per the Pairing Principle), and since there are infinitely many numbers, it will have more unit-parts than any finite number (per Euclid's Axiom).

This formal explication of measure allows us to articulate our third formula. Let k and l each stand for a *measure*, as this was just explicated (and, as before, Fx stands for 'x is finite'); then the formula can be stated as follows:

$$(iii) \exists k \forall l (Fl \rightarrow k > l).$$

This formula states the existence of a measure that is greater than any finite number, though it is not itself a number. With due acknowledgment that Locke lacked the means to fully and adequately develop this idea (for example, a formal treatment of the quantifiers), I believe that we can interpret him as holding that infinite quantities have a measure in the sense captured by (iii)—and that this measure is not a number, contrary to (i). Moreover, since it is not simply an iterative property of an aggregate, this measure is distinct from what is expressed by (ii). My proposal, in sum, is that Locke's notion of quantitative infinity is expressed by (iii) rather than by (i).

By contrast, in the case of Leibniz, there is reason to think that he rejects formula (iii) in addition to (i). Recall what (iii) says: there is a measure—infinite measure—that is *greater than* any finite measure and is *equal to* the measure of other infinite quantities. On Leibniz's view, it seems that these comparisons, between infinite and finite quantities and between different infinite quantities, are ruled out. For, first, once again, infinite aggregates are not wholes, and therefore do not fall under the scope of Euclid's Axiom: it is not the case that such aggregates are *greater than* their finite parts. And second, regarding *equality*, infinite aggregates arguably do not fall under the scope of

the Pairing Principle either.⁵² But such comparisons are preconditions for the applicability of the relevant notion of measure.⁵³

If this is correct, then infinite aggregates do not have a measure according to Leibniz. They thereby do not have an infinite measure either: (iii) is false. Infinity thus understood is not a quantitative notion, of a piece with the notion of a finite measure. Rather, according to Leibniz, to say of an aggregate that it is infinitely large is to describe a certain iterative property of its parts: for any one of them, a greater one exists, as in (ii).⁵⁴

My proposal, then, is that Leibniz accepts (ii), which expresses an iterative notion of infinity, but rejects both (i) *and* (iii): he rejects the notion of an infinite number *and* the notion of infinite measure. Locke also accepts (ii) and rejects (i), the notion of an infinite number. But unlike Leibniz, he accepts (iii), the notion of an infinite measure, which I argued is behind his claim that infinity is ‘a mode of quantity’. The disagreement between Locke and Leibniz can thus be

⁵² As Leibniz writes: ‘There is an actual infinite in the mode of a distributive whole, not of a collective whole. Thus something can be enunciated concerning all numbers, but not collectively. So it can be said that to every even number corresponds its odd number, and vice versa; but *it cannot be accurately said that the multitudes of odds and evens are equal*’ (letter to Des Bosses from 1 September 1706, G 2.315, emphasis added).

⁵³ Levey (2015) suggests that although Leibniz cannot appeal to Euclid’s Axiom in order to secure comparisons of inequality between infinite aggregates and finite aggregates, he can appeal to a different principle: if there is no *injection*, or one-one correspondence, from one aggregate *into* another, then the one is greater than the other. On Levey’s view, furthermore, securing this comparison is crucial for Leibniz’s definition of an infinite aggregate as an aggregate that has more than n elements, for any number n (Levey 2015, p. 184). Despite the appeal of Levey’s suggestion, however, it is not clear that it is supported by textual evidence, since it is not clear that Leibniz endorses an entailment from absence of injection to inequality. Nor is it clear that such an entailment is necessary to maintain Leibniz’s iterative definition of infinity; for this, all that is required is formula (ii).

⁵⁴ Similarly, to speak of the infinitely small is to describe a certain iterative property: for any given small quantity, a smaller quantity exists. This formulation allows Leibniz to perform mathematical operations in his calculus without committing himself to the existence of infinitely small quantities, or infinitesimals, which he treats as ‘fictions’; he writes: ‘I hold that there are no more infinitely small magnitudes than infinitely large ones, that is, that there are no more infinitesimals than infinituples. For I hold both to be fictions of the mind due to an abbreviated manner of speaking....’ (G 2.305). For example, to speak of the sum of an infinite series is an ‘abbreviated manner of speaking’, which can be explicated without appeal to infinitesimals: ‘Whenever it is said that a certain infinite series of numbers has a sum...all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like’ (LoC 99). On Leibniz’s fictionalism, see Ishiguro (1990, ch. 5), Jesseph (1998), Knobloch (2002), and Levey (2008).

seen to be centred on the status of (iii), and in particular on what entities can or cannot be measured.⁵⁵

This interpretation of Leibniz's notion of infinity is consonant with a now standard picture of him as offering a resolution of Galileo's Paradox consistent with Euclid's Axiom. There is a sense in which, by exposing a substantive difference between Leibniz's position and Locke's, the present interpretation goes further. Recall that Galileo's Paradox is generated by two principles of measurement, Euclid's Axiom and the Pairing Principle. Contemporary orthodoxy instructs us to reject the former (and to accept the latter). An important question, recently raised by unorthodox approaches to the set-theoretic notion of measure or size, is why we should prefer the Pairing Principle over Euclid's Axiom.⁵⁶ But perhaps another, more basic question is whether in the case of the infinite we should accept *any* such principles, as opposed to resting content with an iterative property of an aggregate or a set, per formula (ii). Perhaps the lesson to be learned, via contrasting Leibniz with Locke, is that the problem originates in the very notion of infinite measure.

My primary aim in this section has been to address the question, highlighted above, about the relation between *iterative infinity*, which I have argued is associated with formula (ii), and *quantitative infinity*, which I have argued is associated with formula (iii). We can now see that they are indeed distinct from each other, as well as from the notion of *ontic infinity*. If this is correct, then, as anticipated at the outset, we are led to recognize not one, or two, but (at least) three infinities in early modern philosophy.⁵⁷

⁵⁵ As for Descartes, he arguably accepts (ii), with his notion of the indefinite. In addition, as explained in §3, he endorses a non-quantitative, ontic notion of infinity. (Descartes' attitudes towards (i) and (iii) are less clear, and for our purposes can be set aside.)

⁵⁶ See, for example, Mancosu (2009), Mancosu (2015), and Whittle (2015).

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