The Paradox of Sufficient Reason

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> One should think of the paradoxes as supernatural creatures, oracles, minor demons, etc.—on whom one should keep a weather eye in case they make prophecies or otherwise divulge information from another world not obtainable by any other means.
> —T. E. Forster, *Set Theory with a Universal Set*

The Principle of Sufficient Reason, PSR, says that for any truth there is a sufficient reason. Although it has lost much of its luster over the years, lately there has been a revival of interest in this great old jewel of rationalist philosophy. It's not hard to see why. Granted, the arguments against PSR are formidable: it is thought to be subject to philosophical counterexamples, to conflict with the deliverances of quantum mechanics, and to imply the impossibility of contingent truth. Still, a degree of intellectual sympathy for the principle remains. PSR can be interpreted in different ways, but it is perhaps most attractive when understood as saying that any truth, any fact, is explicable. Appeals to explicability are commonplace in science and philosophy, not to mention ordinary life. At least sometimes it is recognizably correct to suppose that a given fact should admit of

Thanks to Christie Thomas, Jeff McDonough, Alyssa Ney, and Michael Della Rocca for comments on various drafts, and particularly to Kenny Walden for comments and advice on nearly every draft. My thanks also to audiences at the 2014 Early Modern Philosophy Workshop in Tahoe City, California, and the 2015 Philosophy Mountain Workshop in Whistler, British Columbia, where earlier versions were presented. I am grateful as well to the editors and two anonymous referees for this journal for all their help.

Philosophical Review, Vol. 125, No. 3, 2016
DOI 10.1215/00318108-3516956

explanation even if we are in no position to provide it. But, as Michael Della Rocca (2010) counsels, once started along this path it is hard to stop. Why should appeals to explicability carry force in some cases and not in others? No satisfactory principle for drawing the line seems forthcoming, however, and so it looks arbitrary to accept one's favorite appeals to explicability while denying PSR when the chips are down. It might as well be admitted too that the intuition PSR codifies still exerts its old pull, as it will always be difficult to regard any given fact as brutely inexplicable, as having simply come from nothing.[1] Also, renewed interest in the idea of "grounding" and a more expansive view of the possible forms of explanation have shifted attention back toward PSR,[2] as have recent studies of PSR in early modern philosophy that look upon it with a critical contemporary eye.[3] It is no surprise, then, to find PSR and the arguments that surround it up for reappraisal. We should want to understand them better than we do; they have more left to teach us.

In this essay, I examine some implications of PSR for contingent truth, particularly as they arise in the context of a celebrated argument due to Peter van Inwagen and Jonathan Bennett.[4] The argument purports to show that PSR entails that there are no contingent truths. Replies to it have been made,[5] but I think a more illuminating response lies in an altogether different direction.[6] The proof at the heart of the van Inwagen-Bennett argument is a paradox that can be used to show that, given PSR, there is no conjunction, or other single totality, of all contingent truths. Accepting this much, the question now becomes an interpretive one: *Quid sibi velit*? Why should there be no totality of all contingent truths if PSR is true? I shall argue that a friend of PSR—at least one who would also allow for the existence of contingent truths—ought to interpret the proof as indicating that there is simply no such thing as

1. Compare Guigon 2015.

2. For instance, see Dasgupta, forthcoming; Schnieder and Steiberg forthcoming.

3. For instance, see Della Rocca 2012 and Lin 2012.

4. See van Inwagen 1983, 202–4; and Bennett 1984, 115.

5. See Vallicella 1997; Pruss 2006; Dasgupta, forthcoming; Guigon 2015; and Schnieder and Steiberg, forthcoming.

6. There are some anticipations. The assumption that I shall call into question—namely, that there exists a conjunction of all contingent truths if there are contingent truths—is mentioned en passant by Oppy (2006, 281) as one that a proponent of PSR will have to reject to avoid the conclusion of the van Inwagen-Bennett argument (Oppy takes this to be a reason to refuse "strong" forms of PSR and to seek more acceptable, weakened versions). The possibility of challenging the assumption is recognized implicitly by Della Rocca (2010, 9n13). Guigon (2015, 356–57) notes it as well.

"all contingent truths." With the proof so taken, PSR can then be seen to involve a quite unexpected picture of the world of contingent truth, and the rationalism that embraces PSR to have a notable affinity with traditional constructivism in philosophy of mathematics.

This will take some explaining. In what follows, I'll first present and take apart the van Inwagen-Bennett argument, exposing the paradox at work within it and identifying the argument's dependence on an auxiliary assumption that a friend to PSR should not accept: if there are any contingent truths, there is a conjunction or totality of all contingent truths. Then drawing on ideas from philosophy of mathematics, I'll formulate two new ways to interpret the paradox, each of which could explain how there might be contingent truths but yet no totality of all contingent truths. Of the two, one—which I'll call the "completist" view—says the completed domain of all contingent truths contains "too many" truths to totalize; the other—the "extensibilist"—says instead that the concept *contingent truth* is indefinitely extensible and rejects the idea of "all contingent truths." I'll develop the rationales for both positions and argue at length that advocates of PSR should favor the extensibilist. This interpretation yields a number of surprising consequences about the commitments of rationalism.

1. PSR, the Paradox, and the Reply in Outline

A sufficient reason both *entails* and *explains* that for which it is a sufficient reason. If R is a sufficient reason for T, then R entails T and R explains T. (I shall sometimes say R is an *explanatory ground* for T.) PSR says that for any T there's always such an R. The van Inwagen-Bennett argument for holding PSR to entail that there are no contingent truths runs as follows.[7]

Let C be the conjunction of all contingent truths. Then C itself is a contingent truth, for no necessary truth can have a contingent truth as a conjunct. By PSR, there is an explanatory ground G that is a sufficient reason for C. G entails C and explains C. Is G itself a contingent truth? If so, then G is in C. But then in explaining C, G would also explain itself, and no contingent truth can explain itself. If G is not a contingent truth but a necessary truth, then because G entails C, it follows that C is a necessary truth, contrary to hypothesis. So, given PSR, there can be no conjunction C of all contingent truths. If there is no conjunction C of all

7. See van Inwagen 1983, 202–4; 2002, 104–7; and Bennett 1984, 115. See also Della Rocca 2010, whose presentation I follow here.

contingent truths, then it must be that there are no contingent truths. Therefore, PSR entails that there are no contingent truths.

If correct, the argument forces a choice between PSR and contingent truth. Either PSR holds and all truths are necessary or else there are contingent truths and some truths are "brute," true but having no explanatory ground, no sufficient reason. This purported necessitarian consequence of PSR is then typically wielded as a sword against it.

In my view, the van Inwagen-Bennett argument makes a powerful case for saying that PSR entails that there is no conjunction of all contingent truths. But it does not show what it purports to show—that PSR entails that there are no contingent truths—and a dose of philosophy of mathematics can help to see why. One lesson especially clear in philosophy of mathematics is that proofs can be interpreted in more than one way. Just what a paradox tells us can be a delicate question, calling for judgment. So it's instructive to see that the van Inwagen-Bennett argument contains both a paradox in the classic style and an interpretation. I'll rearrange its elements to bring this out, presenting the reasoning in four pieces: the paradox, its link to PSR, a logical consequence, and then interpretation.

First, the paradox concerning sufficient reasons—it won't do to have two PSRs, so let us call it *the paradox of the explanatory ground*:

> Let C be the conjunction of all contingent truths. Suppose there is an explanatory ground G for C that explains and entails C. Is G itself in C? Answer: G is in C if and only if it is not.[8] Contradiction.

Second, by PSR, there is such a ground G for C. Elided slightly, the result is this:

> Given PSR, if C is the conjunction of all contingent truths, then there is an explanatory ground G for C such that G is in C iff G is not in C.

The third piece now extracts as a consequence the following conditional—call it the *No C Lemma*:

> Given PSR, there is no conjunction C of all contingent truths.

The fourth piece then adds to the lemma a separate auxiliary assumption:

> If there are any contingent truths, there is a conjunction of all contingent truths.

8. Suppose G is in C. Then G is self-explaining, and so not contingent; so G is *not* in C. Thus, if G is in C, G is not in C. Suppose G is not in C. Since G entails C, if G is not contingent but necessary, then C is not contingent, contrary to hypothesis; so G is contingent; so G *is* in C. Thus if G is not in C, G is in C.

This auxiliary (or its contrapositive) is in effect the principle through which the van Inwagen-Bennett argument interprets the No C Lemma. Why, given PSR, is there no conjunction C of all contingent truths? If C does not exist, this can only mean that there are no contingent truths to conjoin. Hence, PSR entails that there are no contingent truths.

But notice now that the auxiliary assumption is itself the consequence of two unstated presuppositions:

Totality For any contingent truths, there is a single *totality*—in this instance, a *conjunction*—to which they belong.

Completeness If there are any contingent truths, there is such a thing as *all* contingent truths.

Those presuppositions jointly control how the van Inwagen-Bennett argument reasons about the domain of contingent truths and the idea of a single totality containing them all. If Completeness is true, then the case of *all* contingent truths is a case (the absolutely exhaustive one) of *any* contingent truths, and so by Totality, there will be a single totality—the conjunction C—to which all contingent truths belong. Raising the presuppositions to salience identifies two quite different grounds on which the auxiliary assumption might be resisted. Perhaps, contrary to Totality, not just *any* contingent truths can be "totalized" or collected together in a single conjunction. Or perhaps, contrary to Completeness, there can be contingent truths without there being any such thing as *all* contingent truths. Correspondingly, there arise two new interpretations of the No C Lemma to stand as alternatives to that of the van Inwagen-Bennett argument—that is, two potential reasons that could explain why there is no conjunction C of all contingent truths, if PSR is true, that yet need not deny the very existence of contingent truths.

One interpretation (the completist's) would follow a standard account of the set-theoretic paradoxes and seek to reject Totality on the grounds that there are "too many" contingent truths to combine in a single totality.[9] The principle Totality looks like a comprehension principle: for any xs, there is a y such that y contains the xs. Familiar paradoxes

9. The notion of "too many to totalize," in set theory, descends from Cantor's idea of an "absolute infinite" multitude which is "beyond mathematical determination" (Cantor 1932 [1887–88], 405; see also 1932 [1883], 205). "Too many" is also especially associated with von Neumann's (1925) "limitation of size" conception for sets: roughly, if a given class X can be mapped onto the entire set-theoretic universe, then X contains too many members to form a set.

concerning "big totalities"—the universal set, the set of all sets, the set of all ordinals, and so forth—have led to considerable skepticism about such comprehension principles, and their standard resolution involves the idea that sometimes there are "too many" things to totalize. I'll develop this idea further below, but I shall argue that, in the case of the paradox of the explanatory ground, it is not a satisfactory interpretation: it does not adequately explain why PSR entails that there is no such totality as C. I'll also suggest a reason to think that attempts to solve the paradox by denying that just any contingent truths can be combined in a conjunction are looking in the wrong place.

Recent work on the idea of indefinite extensibility in the philosophy of logic and set theory suggests a quite different interpretation of the paradox (the extensibilist's),[10] one that permits a response to the van Inwagen-Bennett argument against PSR that would reject Completeness. This is the interpretation I favor. What the paradox within the van Inwagen-Bennett argument really shows—more properly, what a friend of PSR and contingent truth should take it to show—is that PSR implies that the concept *contingent truth* is indefinitely extensible. Here's the usual formula:

> A concept F is indefinitely extensible iff for any totality T of things all of which satisfy F, one can, by reference to T, identify a further object *x* satisfying F but not belonging to T.

Nicely, it is the van Inwagen-Bennett argument's own proof of the No C Lemma that illustrates how *contingent truth* satisfies this formula. Some fine points will be in order, but for now a sketch suffices.

> Let P be a totality of contingent truths. By PSR, there is an explanatory ground G for P. Now, G cannot be necessary, for then P would be necessary. Nor can G be included in P, for then G would be self-explaining. So, G is a contingent truth not in P. The argument is general, so it holds for any totality of contingent truths. Therefore, the concept *contingent truth* is indefinitely extensible.

10. And not so recent. The idea appears first in Russell 1907 and was championed in Dummett 1963; 1991, 315–19; and 1994. It has lately enjoyed a small renaissance including too many items to survey instructively here. For a start, see the discussion in Shapiro and Wright 2006 of the idea of indefinite extensibility and its development from Russell, and the pieces by Fine and Hellman (and Rayo and Uzquiano's introduction) in the same volume, Rayo and Uzquiano 2006. The present essay is indebted to Velleman 1993 as well. See also Uzquiano 2015a.

This undercuts the van Inwagen-Bennett argument's inference to the claim that PSR implies that there are no contingent truths. Why? If a concept F is indefinitely extensible, it cannot be "exhausted in extension" and there is simply no such thing as all Fs. So if *contingent truth* is indefinitely extensible, then Completeness is incorrect: there can be contingent truths and yet no such thing as all contingent truths and so no conjunction C of all contingent truths. And thus—on the extensibilist interpretation—the van Inwagen-Bennett argument falls through.

2. Two Gaps in the Argument

Now for the fine points. There are two gaps in my argument as stated. One is straightforward to close and holds technical interest. The other takes me to the main philosophical inquiry of the rest of the essay.

First, the sketched proof just given that *contingent truth* is indefinitely extensible only establishes that for any totality P of contingent truths, we can by reference to P show that *there exists some* contingent truth G not in P. What has *not* been shown is that by reference to P we can identify a *particular instance* for G. And it is this latter condition that is usually associated with the definition of 'indefinitely extensible'. It's a vestige of the definition's origin in Dummett's intuitionistic interpretation of the paradoxes of set theory. Russell's Paradox and the Burali-Forti paradox, for instance, yield proofs that there exists no set S of all sets and no set Ω of all ordinal numbers, respectively. Those proofs can be converted into proofs that the concepts *set* and *ordinal* are indefinitely extensible to the letter, identifying particular "further instances."

This matters to Dummett's philosophy of mathematics because the promise that we can single out a specific further instance of the concept in question accords with a principle of intuitionistic logic concerning existential quantification. To say 'there exists an F' requires for its justification the ability to produce an instance. But I don't think this scruple has to be written into the definition of indefinite extensibility. A more neutral formula is possible as well:

> A concept F is indefinitely extensible iff for any totality T of things all of which satisfy F we can show, by reference to T, that there exists some further object *x* that satisfies F but does not belong to T.

On an intuitionistic interpretation, this says the same thing as the original definition, because to say "we can show, by reference to T, that there exists some further object *x*…" is, for the intuitionist, just to say "we can, by reference to T, identify a specific further case *k*. . . " On the usual (classical) interpretation of the quantifier, however, we are permitted to assert the general claim "there exists some further object *x*. . . " without thereby claiming to be able to identify any specific case. Severing the explicit demand for a particular instance from the definition leaves the old dispute between classical and intuitionistic logic intact while also allowing each side a formula for indefinite extensibility.

With the neutral definition of 'indefinitely extensible', the sketched proof counts as a proof proper that the concept *contingent truth* is indefinitely extensible. This still distinguishes the proofs in the mathematical cases from the proof in the PSR case: the former are intuitionistically valid while the latter is only classically valid. But it also allows us to see the tight connection between them as not merely an affinity: they are one and all proofs that a given concept is indefinitely extensible. If it should still be objected that the neutral reformulation changes the subject, infringing on Dummett's copyright on the term, we can drop the label 'indefinitely extensible' and focus just on the logical properties at hand. The proof establishes that for any totality of contingent truths P, there will always be a further contingent truth G not contained in P. And that suffices to set the stage for the interpretation of the paradox that rejects Completeness.

Now to the second gap. The indefinite extensibility of *contingent truth* does not strictly imply the falsity of Completeness. If a concept F is indefinitely extensible, it follows only that it cannot be exhausted in a single totality like a conjunction or a set. This does not yet rule out the formal possibility of holding F to be exhaustible in extension in *its instances*—that is, in absolutely all the Fs taken "uncollected" as a plurality. The Fs might not all fit into a single set, but they might all still *exist.* This gap could be closed by stipulating what Richard Cartwright (1994) skeptically calls the "All-in-One Principle": that the domain for quantification must always be a single set or totality. If there is no such totality of Fs, then 'all Fs' has no clear meaning. But the All-in-One Principle is controversial. And, like Cartwright, I don't accept it.

An alternative maneuver might be to revise the definition of 'indefinitely extensible' again in a slightly different way by replacing its reference to a single totality T with plural reference to the elements themselves. The amended plural form would then say: a concept F is indefinitely extensible if and only if for any *y*s each of which satisfies F, one can by reference to the *y*s show that there exists a further object *x* that satisfies F but is not one of the *y*s. So formulated, it appears straightforward that an indefinitely extensible concept cannot be understood to be exhausted in extension, whether in a set or in its instances.

I am sympathetic to this plural definition, as it captures a natural idea of indefinite extensibility and has some advantages of its own.[11] But while it would nicely close the gap between indefinite extensibility of *contingent truth* and the denial of Completeness, it would open a different one in the present argument. The sketched proof of the indefinite extensibility of *contingent truth* would not show that the concept satisfies the plural definition, for the proof relies directly on reference to a given single conjunction P of contingent truths. PSR will say of any such P that it must have an explanatory ground G, which can then in turn be shown by the paradox not to belong to P. If instead we bypass P altogether and look only to the individual contingent truths themselves, there is no promise that PSR will require a single explanatory ground G for all those truths taken unconjoined. And of course without such a G, the claim that there is a further contingent truth outside of those already acknowledged—and with it the claim of indefinite extensibility—disappears. So in this context the plural definition won't provide a workable replacement.

One formal principle already articulated will of course close the gap in argument: Totality. If any contingent truths form a totality, and if, as the sketched proof shows, for any totality of contingent truths there is provably a contingent truth lying outside of it, it follows immediately that there can be no such thing as all contingent truths. This is no surprise. The paradox already told us that PSR, contingent truth, Totality, and Completeness jointly yield a contradiction. The sketched proof of the indefinite extensibility of *contingent truth* accepts the consistency of PSR and contingent truth, so it's a foregone conclusion now that adding Totality will yield a disproof of Completeness. Still, it is not clear to me whether an extensibilist defender of PSR should want to insist on Totality. There is of course the "bad company" worry: similar-seeming comprehension principles have demonstrably come to grief in set theory, so caution is in order. Also, it seems to me that weaker assumptions more naturally related to PSR and contingent truth might provide grounds for rejecting Completeness while remaining neutral about Totality. I'll explain my reasons for this shortly.

11. See Uzquiano 2015a.

In any case, even if some technical device might secure a quick inference from the indefinite extensibility of *contingent truth* to the idea that there is no such thing as all contingent truths, I think it will be more illuminating to close the gap not with a formal stipulation but by providing an informal philosophical rationale that explains both why the concept *contingent truth* is indefinitely extensible and why it cannot be exhausted in extension tout court—whether in a totality or in its instances—if PSR is true. This is just to interpret the paradox of the explanatory ground and the No C Lemma. What are they telling us?[12]

3. The Extensibilist's Rationale

The paradox proves that there can be no explanatory ground G for a conjunction C of all contingent truths. PSR says that any truth has an explanatory ground. So by PSR, it follows *that* there is no such truth as C. But *why* not?

On the extensibilist interpretation, the answer lies in the explanatory demand embedded in PSR and the conception it yields of contingent truth. PSR sees contingent truths as contingent *explananda.* Maybe a necessary truth could be self-explaining, but a contingent truth always has an explanatory ground that lies outside of itself. Also, the explanatory ground must itself be contingent and so equally falls under PSR's demand for a further ground, and so on. An unending explanatory order of contingent truths reveals itself in this way, what we might call a contingent rational order. Now PSR does not say that for just any truths at all there is a common explanatory ground. But for contingent truths that fall into such a rational order, it is very natural to think that we can step back from the order itself and ask why this whole sequence obtains in the first place. It's contingent after all. Why does nature include it rather than not? Such contingent rational orders thus appear to point outside themselves toward further explanatory grounds—which will likewise now be

12. A caveat. I take as my starting point the paradox and the related proof of the lemma. There are of course significant questions one might raise here: for example, whether PSR must be understood to require sufficient reasons to necessitate what they explain (see, by way of comparison, Leibniz 1875–90, 2:56, and Pruss 2006, 104ff.), or whether, even given PSR, a conjunction must have a sufficient reason distinct from the sufficient reasons for its individual conjuncts (see Hume 1947 [1779], 9.9, and similarly Vallicella 1997 and Schnieder and Steiberg, forthcoming). My analysis will not be questioning those assumptions. The proof of the No C Lemma strikes me as sufficiently natural and compelling to make it worthwhile to try to understand the lemma while leaving the proof intact.

contingent and come under PSR's explanatory demand, setting up a further progression. And then we can step back again from this extended sequence, and so on.

In this way, I think, it is clear that the rule for identifying explanatory grounds is inexhaustible when applied to contingent truths. I'll fill out this picture in more detail later. For now the lesson is this. From the point of view of PSR, the concept *contingent truth* is not only indefinitely extensible, it's not "extensionalizable." Demonstrably no such order can be complete: given PSR, by reference to any contingent truths belonging to such an order, it can be shown that there is a further contingent truth not among them but also falling under its same rule for explanatory grounds and so extending that order. No such contingent rational order can be complete—there can be no such thing as "all the contingent truths" contained in it—and so neither can the general domain of contingent truths be complete. That's why PSR entails that there is no conjunction C of all contingent truths and the falsity of Completeness.

Observe now that this informal rationale does not appear to invoke the broad principle Totality, that just any contingent truths will form a totality. It suffices that for any contingent truth there will be a contingent rational order into which it falls, and that for any such order, it makes sense to ask for its explanatory ground. Perhaps this means taking any rational order to be a totality, but it does not amount to saying that any contingent truths *whatsoever* form a totality. So assumptions weaker (about totality formation, anyway) than Totality are enough to fill out the argument against the idea of all contingent truths.

As noted, though, this extensibilist rationale is not mandatory. A competing interpretation of the No C Lemma is possible that models itself on the historically more standard approach to the paradoxes of set theory. This rival, the completist account, allows for *contingent truth* to be completely exhausted in extension and explains the nonexistence of C by saying that there are too many contingent truths to combine in a single totality. It thus rejects Totality.

The distinction between Totality and Completeness points to different kinds of concerns raised by the paradox. The assumption of Totality is that there are no restrictions on combining contingent truths together. Its falsity would indicate that there are limits on what it is, or what it means, for something to be such a single totality. By contrast, the assumption coded in Completeness is that there is a completed domain, an exhaustive final inventory, of contingent truths. Its falsity would mean that the very idea of absolutely all contingent truths is not coherent.

The extensibilist and the completist interpretations can each block the van Inwagen-Bennett argument by rejecting one of its key presuppositions. But their informal rationales are at odds. (They correspond approximately to "constructivist" and standard "classical realist" outlooks in philosophy of mathematics.) So a choice has to be made. I'll suggest reasons specific to the case of the No C Lemma for favoring the extensibilist. But it's also useful to develop the contrast between extensibilist and completist views in general terms, and this can be brought out with extra clarity by looking at a parallel contrast in the interpretation of the paradoxes of set theory. I'll consider just one example, not too formally but in some detail.

Take the ordinal numbers ($0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega+\omega, \ldots, \omega^2, \ldots, \omega^\omega$, etc.). Ordinals describe order-types of well-ordered sequences. If the counting numbers are defined by the "successor" operation, ordinals are counting numbers supplemented with the operation of "taking limits" of infinite sequences and with the idea that an ordinal is the set of its predecessors. Any ordinal is the well-ordered set of all smaller ordinals: it includes them and describes their natural order. Precisely defined (in von Neumann's style): a set O is an ordinal just in case it is strictly well ordered by set membership and every element of O is a subset of O.

There are ordinal numbers—many, *many* of them—and each is itself a set of ordinals. But there can be no totality, or set, of all ordinals. This is proved by the Burali-Forti paradox.

> Let Ω be the set of all ordinals. Then Ω is strictly well ordered by set membership, and every element of Ω is a subset of Ω. (Think about it.) So, by the definition of ordinal, Ω itself is an ordinal, and it follows that Ω contains itself. But if $\Omega \in \Omega$, then $\Omega < \Omega$, which is a contradiction. So there is no set Ω of all ordinals.

Why is there no such set as Ω? The extensibilist and the completist answers diverge sharply, bringing their philosophical contrast into high relief. Start with the extensibilist.

Note first that the disproof of Ω yielded by the Burali-Forti paradox can be converted into a proof that the concept *ordinal* is indefinitely extensible.

> Suppose O is a well-ordered set of ordinals. Then by the usual definition, O itself is an ordinal and is larger than any ordinal it contains. If O is in O, then $O < O$, which is a contradiction. Thus O is not in O, but O is an ordinal. Likewise for any such set of ordinals: by reference to it we can find a further ordinal not included in it.

Indefinite extensibility implies that *ordinal* cannot be exhausted in a single set. And for this phenomenon, the extensibilist can offer a natural informal rationale. The concept *ordinal* embeds rules and procedures for defining ordinals. It is given to us precisely by our acquired grasp of the interplay between the operations "successor," "taking limits," and "set of." And by virtue of that grasp, we see that the procedures for defining ordinals, by their very nature, cannot be exhausted tout court.

Think of it this way. Take the ordinals corresponding to the counting numbers 0, 1, 2, etc., letting the "successor" operation play all the way out. The "set of" and "taking limits" operations then allow one to step back and look across the whole sequence so far defined and to codify this new perspective in a single collection, yielding a new ordinal, ω. And then the process starts again with "successor," etc. Can *this* process play all the way out, exhausting the rule for defining ordinals? Provably it cannot play all the way out to result in some single set. Can it play all the way out to result in *absolutely all the ordinals*, as it were, the completed output of the three interlocking operations?

If the concept *ordinal* is given to us through the rule for defining ordinals, it is hard to understand there being "absolutely all ordinals." Here's why. The rule delivers ordinals in an obvious natural order. If absolutely all ordinals exist as the completed output of the rule, it seems that we should be able to ask what the order-type is of their completed natural ordering. But since order-types are given by ordinals, this leads straightaway into the Burali-Forti paradox. The conception of the ordinals as completed makes the question about order-types of sequences of ordinals unintelligible precisely in the imagined exhaustive case.

By contrast, if the rule for defining ordinals is inexhaustible, then for any ordinals you like, the question "What is the order-type of their natural ordering?" always yields a clear, coherent answer. That answer is given from a perspective that is immediately codified in a further ordinal, giving rise to a more extensive sequence of ordinals, and so on. The idea of getting outside all ordinals, once and for all, to consider their order fails to recognize that the "stepping back" is just another extension of the operations for defining ordinals. Or so says the extensibilist.

On the other side, the completist regards the ordinals as completed in extension. Though perhaps *defined* by the operational rules, their great sequence *consists* in the ordinals themselves, absolutely all of them. To avoid the Burali-Forti paradox, the completist introduces a novel distinction between sequences of ordinals that can be collected together in a set and sequences that are "uncollectable." Any proper initial sequence of the ordinals is collectable into a set, and its natural order has an order-type; the entire sequence is uncollectable, and its natural order has no order-type.

The new distinction allows the completist to describe the sequence of ordinals consistently, but it does not answer the question of *why* some sequences of ordinals are collectable and others are not, why some have order-types and others do not. And this is just the original question of why there is no set Ω of all ordinals. Here, however, the completist is out of answers and has to say it's just a basic fact about ordinals. The completist shorthand is to say that there are too many ordinals to fit into a single set, too many for their natural order to have an order-type. But the distinctions between 'collectable' and 'uncollectable', 'not too many' and 'too many', appear only to be pointing to the Burali-Forti paradox—to "wield the big stick," in Dummett's phrase (1991, 316)—not giving an explanation.[13]

In general, the hope of the extensibilist line of interpretation of paradoxes like Burali-Forti is that by placing the explanatory burden on the content of the concepts themselves, we can form some positive grasp of the reasons why there can be no totality of all ordinals (and likewise no set of all sets, no universal set, and so forth). The suggestion is that we come to understand the concepts by coming to grasp rules for constructing, or defining, instances, and through this we see that there can be no end to the process. It must be admitted that this is enormously plausible as a description of both the protocol and the experience of mathematical training. We discover what the concepts involve by learning the rules and procedures for their deployment.

This does not have to mean that we never transcend that procedural understanding and form a further conception of fixed domains whose elements fully exhaust the concepts, as well as a grasp of the novel distinctions imagined by the completist. A defender of the completist view is free to try to explain this further conception. The advantage of

13. There's a further difficulty with interpreting Burali-Forti in terms of 'too many to totalize'. As Hellman (2011) nicely shows, the account of ordinals and the Burali-Forti paradox itself can be cast in plural terms. Since this bypasses reference to totalities of ordinals, the paradox cannot be solved just by saying that there are too many ordinals to totalize. It appears the completist must instead say that there are too many ordinals to well-order, despite their obvious natural order. Alternatively, pluralized Burali-Forti can be interpreted as a straightforward proof that there is no such thing as "absolutely all ordinals." And this can in turn be converted into a proof of the indefinite extensibility of *ordinal*.

the extensibilist approach now is that it can dispense with this whole second tier of explanatory work and simply put its stock in the rules and procedures of mathematical practice that will inevitably show up in any plausible account of what we understand when we understand mathematics. No *more* transcendent view is required than that to make sense of the paradoxes and their origin. Or, again, so says the extensibilist.

It's not necessary to adjudicate the general dispute between the extensibilist and the completist in interpreting the paradoxes of set theory. It's enough to see the comparative pull that extensibilism will have over completism for one who finds PSR compelling. A completist may feel that I have not presented the two sides altogether evenhandedly in illustrating the rationale for the extensibilist diagnosis. Even so, the favorable light cast on extensibilism in the general case will not constitute the full grounds for preferring extensibilism in interpreting the No C Lemma. Facts specific to that instance will underpin my argument there.

For the moment, though, let me draw out the general extensibilist parallels for PSR and contingent truth. Granted, the analogy is not perfect. As noted already, PSR does not give a rule for precisely defining a *particular* explanatory ground for any given contingent truth or totality of truths. (A sufficient reason for a given truth T cannot even be defined impredicatively as "*the* explanatory ground for T" since PSR, as stated, does not imply that a given truth will have a unique sufficient reason.) It's not a mathematical operation of that sort. Nonetheless, the story is strikingly similar. PSR sees in the concept *contingent truth* a rule for postulating explanatory grounds that can never be exhausted. Stepping back from any contingent truth or totality of contingent truths to find an explanatory ground can lead only to further contingent truths, and thus to further applications of the rule. And because it seems that there is no final limit to this "stepping back," no shift that would lead outside the framework for positing explanatory grounds, there is likewise no clear conception of there being such a thing as once-for-all all contingent truths—all contingent explananda—from which to step back.

By contrast, now, the completist about contingent truth embraces the existence of all contingent truths. And so the completist is stuck needing to explain why it is that for any contingent truth, including conjunctions of contingent truths, we can ask for an explanation, but that for the exhaustive totality of *all* contingent truths this becomes unintelligible. If all those truths exist, why not take their totality C and ask for its explanatory ground? Doing so, however, leads straightaway into the paradox of the explanatory ground. To avoid contradiction, the completist will have to introduce a novel distinction between those pluralities of contingent truths that can be collected into a single totality and those that are uncollectable. Perhaps there are just too many contingent truths to conjoin or totalize.

Again, this looks like an unappealing partnership for PSR, since the crucial distinctions between 'collectable' and 'uncollectable', 'not too many' and 'too many', are left unexplained. But even if some general account of those distinctions were forthcoming to render them explicable, this would provide only a template for claiming that there is no totality, or conjunction, C of all contingent truths. Still needed is a reason to think that there *are* so many contingent truths if PSR is true. What might it be?

4. Too Many Contingent Truths?

One reason for claiming that there are too many contingent truths to totalize might simply be that the idea of an infinite conjunction, as C would presumably be, is inadmissible given the elementary rules concerning conjunction. But that's a minor hitch involving dispensable apparatus. Replace 'truth' and the relation of 'conjunction' with 'obtaining state of affairs' and the more permissive relation of 'inclusion', for example, and recast the argument in those terms.[14]

Deeper potential reasons may be found in the set-theoretic worry about big totalities. Let's try to play the "too many" card in the fashion of the completist approach to set theory. Perhaps there are as many contingent truths as there are ordinals, and, if so, that would be a reason for denying that they can form a totality like C. Why might there be so many contingent truths?

Any answer along these lines will have to indulge in set theory if the claim of "too many" is even to be articulated.[15] Not all indulgences are equal. Here I'll consider three different forms such an answer might take. None, I think, yields a satisfactory explanation of the No C Lemma.

14. Because "inclusion" is as permissive as set-theoretic membership, there is no initial problem in defining C as including infinitely many states of affairs. In a more recent presentation, van Inwagen states the argument in terms of sets of propositions, which likewise removes the hitch (see van Inwagen 2002, 104–7).

15. Note that ordinary logical mechanisms for compounding truths together—arbitrary disjunction, say, or weakening—might yield infinitely many contingent truths (including perhaps individual "infinite" compounds), but they won't yield "ordinal many" truths without assuming either a substantial fragment of set theory or an initial supply of contingent truths as large as the ordinals.

First, set theory might be used from the *outside* to describe the domain of contingent matters, and it may then disclose an unexpectedly vast supply of contingent truths. For example, Alexander Pruss (2006, 100) suggests—taking a cue from David Lewis's "Principle of Recombination"—that for every cardinality *k*, it is possible that *k* be the cardinality of objects in the world.[16] Then for each *k* distinct from the actual cardinality of objects, there is the contingent truth T_k that the cardinality of objects is not *k*. Since there are "ordinal many" cardinalities, there would then be too many contingent truths T_k to totalize.

This invocation of set theory is fair play. How many objects could there be? There's nothing out of order in using powerful mathematical tools for analyzing what 'How many?' may mean and thus what answers it could intelligibly have. Still, is the Lewisian ontological principle true? Similar principles have been articulated for "urelements," the faceless nonset elements introduced in certain versions of set theory: *k* urelements could exist for any cardinality *k*. (Or even: there could be urelements such that there are more than *k* urelements for any *k*.) Perhaps a contemporary Pythagorean who likens all things to numbers would assimilate ordinary objects to urelements and hold that any model in set theory automatically corresponds to an ontological possibility for real objects. But a conceptual gap separates the two. Urelements, being conceptual blanks, place no constraints on their possible coexistence. By contrast, the ordinary idea of real objects—more precisely, of "concrete" objects, objects for which the question "How many?" can clearly have different possible answers—would seem to involve existence in space and some degree of mutual spatial exclusion, and not just any model of urelements makes sense so-interpreted. Paring away those constraining features of the ordinary idea of objects could help close the gap, but only to call into question whether the resulting thin notion really suffices to represent possibility for concrete things.[17] Short of Pythagoreanism itself, it is hard to see what grounds there are for accepting unlimitedly

16. Lewis's original principle is that for any cardinality *k* and any possible world *w*, it is possible that there be *k* duplicates of the objects that exist at *w* (see Lewis 1986, 88–89).

17. Notably, Lewis (1986, 90) adds to his Principle of Recombination the caveat "size and shape permitting." Spatial exclusion could also be due to a less direct constraint. For instance, infinite energy density appears to be inconsistent with physical theory at a very general level; indiscernible bosons might occupy the same quantum space, but only finitely many. It is not clear that an upper bound is a merely contingent feature of the laws, and it's hard to see what concept of concrete object is left without the concept of energy. For a more permissive view, see Hawthorne and Uzquiano 2011, 56.

strong Lewisian correlates to such mathematical principles concerning urelements.

My own initial sense is that, leaving sets on the outside, the universe of contingent possibility, even if enormous, should be small compared to the universe of set theory (as it is standardly understood).[18] *Everything* apart from set theory should be small by comparison. Here I'm inclined to agree with Russell's (1956, 250) quip about Cantor's Theorem:[19] "There are fewer things in heaven and earth than are dreamt of in our philosophy."

Taking heed of the trouble with this first attempt, a second effort to use set theory to explain the No C Lemma might be to posit set-generative principles *inside* the domain of contingent matters, thereby yielding hybrid set-theoretic constructions. For instance, suppose sets with contingent objects as elements are themselves contingent objects. Let some ordinary contingent object be the empty set (or just some urelement) and hold to the usual axioms of set theory so-interpreted. This yields "ordinal many" set-theoretic contingent objects with scant effort. For each such object α, there is the unique contingent truth T_α that α exists, and so too many contingent truths to totalize.

Still more easily, we might dispense with the "objects" and mix set theory with contingent truth via logical construction. For example, for each ordinal number i, there is the truth T_i that i has a successor. Let P be a contingent truth. The conjunction T_i & P, for each i, will also be a

18. If more is asked for my sense of the relative smallness of contingent possibility in particular, I might offer the following: Contingent matters boil down to the existence and arrangement of concrete things, and my grip on this idea is mediated by the thought that such things exist in space. Let the basic elements be no bigger than points. Still, space itself intuitively forms a continuum (or something like it that is sufficiently "connected") and that puts limits on occupancy, though enormously generous limits: it can accommodate $c = 2^{\aleph_0}$ punctile beings. Even if each point should secretly turn out to be a further continuum in its own right, the same limits apply. Adding "arrangements" we top out at 2^c contingent entities. A truly vast domain but still dwarfed by the universe of set theory. If one insists on more possible cardinalities for contingent objects—*far* more, before even approaching "ordinal many"—I lose my grip on the "space" in which so many things are supposed to exist in common and no longer really understand the proposal. Similarly, an advocate of the possibility of unlimitedly many distinct concrete objects being colocated in a single region of space (see Hawthorne and Uzquiano 2011 on "angels") can resist the argument, though on this hypothesis I find myself losing my grip on 'distinct concrete object'.

19. Namely, that the "power set" (set of all subsets) of any given set A is larger than A itself, written $|\wp A| > |A|$.

contingent truth. (Likewise: $T_i \supset P$, for each *i*.) Thus "ordinal many" contingent truths, etc.

We can spin these out all day. But such labor-saving mechanisms for fabricating contingent truths yield only correspondingly "easy" explanations. Injecting the marrow of set theory into *any* subject domain will generate a supply of chimeras too many to totalize. The easy explanations lack domain specificity. Their story of abundance is always the same, the chimeras doing all the work. Note how the largeness of the purported supplies of contingent truths is not due to the contingent parts of those truths, and the contingency of the truths in the large supplies is not due to what makes them large. The relation drawn between 'too many' and 'contingent truth' seems to be one of indifference rather than illumination. But even if one were to look more favorably upon the easy explanations of why there should be too many contingent truths to totalize—perhaps regarding the chimeras as having a stronger specific claim to the contingent realm than I have suggested—there remains a deeper problem.

Recall that the explanatory task is not just to explain why there is no totality C of all contingent truths. It's to explain why PSR entails that there is no such totality, the No C Lemma. The lemma rests on the paradox of the explanatory ground, and plainly PSR's idea of an explanatory ground and the notion of contingent truth are central players in the paradox—key elements salient in the "set up" for the No C Lemma and for the explanatory question about its philosophical significance.[20] Minimally, a satisfactory explanation of the lemma should honor this by connecting the reason for C's nonexistence in an integral way with contingent truth and PSR. Perhaps C could be proved contradictory on completely different grounds. But this would not explain the No C Lemma, or not very well. (Suppose you find a beautiful proof that P implies the falsity of T. I prove by unrelated means that T implies Q & ∼ Q. I can "predict" that P implies the falsity of T by supposing P as an initial premise and then introducing my proof. I have not thereby explained your result.)

20. Consider Lange's (2014) account of the distinction between an explanatory proof of a mathematical theorem and a "brute force" proof that reaches the same result without being genuinely explanatory. It seems apt for the present case of the No C Lemma as well: easy explanations do not exploit the features of the "setup" of the problem that are the salient features of the lemma (PSR, contingent truth, and so forth); they operate by brute force via the generative mechanisms of set theory.

As readily comes to light, though, rationales that look primarily to set theory itself to underwrite the claim of too many contingent truths—like those just surveyed—have a hard time meeting the minimal constraint of connecting 'too many' and 'contingent truth' in an integral way with each other and with PSR. The set-theoretic constructions crucial to those rationales appear simply to be extraneous to the No C Lemma. The proof of the lemma makes no assumptions about the existence of such constructions, and its reasoning would appear to remain intact even if their existence were denied. The reasons offered by the easy explanations aren't addressing the central phenomena of the paradox, and so aren't getting to the heart of the matter. Is there a better prospect?

Consider a third approach. This would be to argue from first principles that set theory belongs to "fundamental ontology" and that the basic relationship between set theory and the theory of propositions guarantees the existence of too many contingent truths. Here I have in mind a strategy that could invoke Russell's (1903, 527–28) paradox of propositions from appendix B, section 500, of *The Principles of Mathematics.* Here is a simple version in terms of sets.

> Suppose there is a set P of all propositions. Take all the subsets S of P. Each subset S is a set of propositions, and it seems that, for each one, there is a uniquely corresponding characteristic proposition p_S that says "Every element of S is true." For each S and its characteristic proposition p_S, either $p_S \in S$ or $p_S \notin S$. Now consider the set w of just those propositions p_S such that $p_S \notin S$, that is, $w = \{x \subseteq P \mid p_x \notin x\}$. This set w has its own characteristic proposition p_w, saying "Every element of w is true." Now, is $p_w \in w$ or not? Answer: $p_w \in w$ iff $p_w \notin w$. Contradiction.

It is natural to interpret the paradox as showing that there is no set of all propositions, and a set fundamentalist may take this in turn as indicating that there are too many propositions to totalize. Now observe that if 'proposition' is replaced with 'contingent truth' throughout the argument, the result is a proof that there cannot be a set of all contingent truths.[21] So it might equally be concluded that there are too many contingent truths to totalize. Does this Russellian argument succeed where the easy explanations did not?

Despite its fine credentials, the Russellian argument has much the same diagnosis. "Most" of the contingent truths it identifies are

21. This follows Menzel's (2012) tactic of using Russell's argument to show that there is no set of all *true* propositions.

truths about sets, not about first-order contingent things like apples and raindrops. The great edifice it constructs consists of truths saying "Every element of set S is a contingent truth" for sets S whose elements are truths that themselves say the same thing about further such sets, and so on—needing no more than a single ordinary contingent truth as its base. Another easy fabrication mechanism, if highly glorified. Contingent truth is little more than a token in the argument, and PSR's idea of an explanatory ground is given no role at all. The Russellian argument does not help to explain the No C Lemma and the underlying paradox. It would, at most, corroborate the claim that there is no such totality as C on different grounds.

The set fundamentalist here might seek to overrule my minimal constraint on a satisfactory explanation. If fundamental ontology dictates that there's no such totality as C, that's explanation enough. What to say? I'm not particularly moved by this. As a side note, I think it's fair to be skeptical of set theory's claim to being fundamental in ontology, as opposed to its just being fundamental in mathematics. And in any case, I doubt whether fundamental ontology ought to bully us out of pursuing an ordinary philosophical explanation of the No C Lemma. Perhaps the lemma reflects some kind of cognitive illusion and needs to be explained *away* rather than explained. Still, even a debunking explanation should connect with the terms of the explanandum.

There is also a different rejoinder open, not to a neutral party perhaps, but certainly to the extensibilist. The Russellian argument can be interpreted in an altogether different way: as a proof that the concept *contingent truth* is indefinitely extensible.[22] Thus even granting set funda-

22. Let P be a totality of propositions satisfying the concept *contingent truth.* Take all the subsets S of P. To each such subset S there corresponds its characteristic proposition p_S. If those characteristic propositions are all elements of P, then there is a subset R of P of exactly those propositions p_S that are not elements of the set S to which they correspond. That is, $R = \{x \subseteq P \mid p_x \notin x\}$. But then R's characteristic proposition p_R will be such that $p_R \in R$ iff $p_R \notin R$, which is a contradiction. So, there is no such subset R of P. And hence the characteristic propositions p_S cannot all be elements of P; rather, at least some must lie outside of it. Further, each characteristic proposition p_S itself also satisfies the concept *contingent truth.* (In each case what p_S says of its S is that every element in S is true. And in each case, that's true. But only contingently so, for the elements of S are themselves only contingently true.) Hence, *contingent truth* is indefinitely extensible. Note that the proof does not show that we can, by reference to any totality P of contingent truths, identify any *particular* contingent truth that falls outside of P, but only that we can identify a set of specific contingent truths at least *some* of which must do so. Thus the neutral formulation of 'indefinitely extensible' is needed to claim the intended result by name.

mentalism, the Russellian argument can be accepted without any appeal to 'too many' and so without falsifying Totality.[23]

This capture of completist arms by extensibilism can be carried further. The whole idea that there are too many ordinals to totalize depends on the Burali-Forti paradox. And as noted, that too can be interpreted in terms of indefinite extensibility of *ordinal* with no appeal to 'too many'. Interestingly, Russell conjectured it "probable" that for any property φ that determines a "self-reproductive class"—his term for indefinite extensibility—"we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the property φ" (1907, 36).[24] In short, for any indefinitely extensible concept, it can be shown to extend across "ordinal many" instances. If true, this would certainly yield a reason to say that there are "ordinal many" contingent truths. But it would then be indefinite extensibility that explains why there are so many contingent truths, and the completist's intended inference to '*too* many' would not go through.[25] (Alternatively: one could allow that there are, in a sense, too many contingent truths to totalize, but now the meaning of 'too many' would be given in terms of inexhaustibility of the concept, and the reason why there is no such totality as C would ultimately depend on the extensibilist grounds for rejecting Completeness.)

Case by case, the attempts to explain why PSR should entail that there are too many contingent truths to totalize appear inadequate—either unjustified or unilluminating or both—and so the tactic of denying Totality for contingent truth in order to escape the van Inwagen-Bennett argument seems to fall short. At this point, let me suggest a reason for thinking that the basic effort to understand the No C Lemma primarily in terms of the falsity of Totality may be misguided.

23. There is a further difficulty with interpreting the result of the Russellian argument in terms of 'too many'. The basic proof can be formulated not only for sets of propositions (and indeed for 'classes' more generally) but also in plural terms that make no assumptions about sets or other totalities (see Uzquiano 2015b). Likewise for the replacement version with 'contingent truth'. To claim that there are too many contingent truths to totalize no longer helps the completist to escape the paradox, since totalities play no role in the plural version. By contrast, the extensibilist can take the plural version as a straightforward proof that there is no such thing as all contingent truths. And this can in turn be converted into a proof of the indefinite extensibility of *contingent truth*.

24. For some discussion see Shapiro and Wright 2006.

25. Though of course an extensibilist would demur at Russell's phrase 'all ordinals'.

The No C Lemma rests on the paradox of the explanatory ground. The paradox shows that there can be no explanatory ground G for the totality C of all contingent truths. I suspect that the totality isn't really the problem. For the paradox, or one almost exactly like it, can be stated without assuming that there is a totality C of all contingent truths.

> Let the Cs be the contingent truths—absolutely all of them. Suppose G is an explanatory ground for the Cs. That is, G entails and explains the Cs. Is G one of the Cs? Answer: G is one of the Cs iff G is not one of the Cs (by the same reasoning as before). Contradiction.

So there's a contradiction in the idea of there being an explanatory ground for absolutely all contingent truths, whether they are conjoined in a totality or not. The paradox cannot be resolved by denying that the Cs form a single totality, since that's not a premise. Likewise for Totality.

This does not automatically mean that C and Totality are red herrings in the inquiry into the No C Lemma. As noted before, it's not clear that PSR should require there to be a single explanatory ground for all contingent truths taken unconjoined. So perhaps in order to link PSR to the paradox of the explanatory ground, something akin to C and Totality will be needed (which of course returns the question to the same case-by-case problems of trying to explain why there should be too many contingent truths to totalize if PSR is true). Still, I think the "unconjoined" version of the paradox suggests that the true source of the difficulty lies elsewhere. And since the idea of there being absolutely all contingent truths is essential to both versions of the paradox, that's some reason—at least for defenders of PSR—to think it's the real culprit. If good sense can be made of the idea of resisting Completeness, then it appears that the paradox of the explanatory ground places no particular pressure on a defender of PSR to deny Totality. If there is no such thing as *all* contingent truths, the putative totality C will never arise, and thus, so far as this paradox is concerned, the question of whether any given contingent truths will always form a totality appears to be an independent matter.

5. PSR, Explanatory Demand, and Indefinite Extensibility

The proposed reply to the van Inwagen-Bennett argument invokes the well-oiled, if not always familiar, machinery of an extensibilist line of interpretation of the classic paradoxes of set theory. In light of the similarity of the paradox of the explanatory ground to the paradoxes of set theory, it should now be no surprise. Unlike the rationales considered for the completist's "too many" interpretation of the No C Lemma, the extensibilist's rationale obviously meets the minimal constraint of explaining the Lemma in a way that integrally connects the key terms. Since the proof of the indefinite extensibility of *contingent truth* is a direct reworking of the proof of the No C Lemma, this was always going to be a virtue for it to claim. And moreover, the extensibilist's rationale is a natural view to take of the explanatory demand embedded in PSR in the first place, even independently of the paradox. This picture can now be filled out further.

As before, PSR trades in part on the intuition that any given thing ought to be explicable and the native difficulty of seeing anything as brutely inexplicable. And of course the concept of contingency is especially subject to the explanationist impulse: contingent truths easily appear as contingent explananda—if something is so but could have been otherwise, it's entirely natural to ask why it *is* so—thus setting the search for explanatory grounds into motion. Part of the appeal of the van Inwagen-Bennett argument's claim that PSR entails that no truths are contingent lies in the inchoate sense that PSR will drive explanation on inexhaustibly unless it finally comes to an end in some self-explaining ground. Small wonder that "rationalists" like Descartes and Leibniz, for instance, put the contingent world onto the shoulders of a necessary being as its metaphysical foundation.

Let me head off a misunderstanding. The perplexity arising from the combination of PSR and contingent truth is not that unending regresses of explanation are unacceptable. Actually infinite regresses of contingent explanatory grounds may be admitted. They just are not enough to satisfy the explanatory demand. Leibniz considers the model of an endless sequence of books on the elements of geometry, each copied from a prior one and thereby having its existence in the series explained. No book is without an explanation. Are we at the end of the explanatory task? No, says Leibniz. For we can step back from the whole series itself and ask why a series of books on *geometry*. Or why a series of books at all.[26] Even an actually infinite totality of contingent explanations will once again point outside itself toward a further explanatory ground.

I think what we find here is that the explanatory demand of PSR with respect to contingent truth is inexhaustible in an exquisitely strong way: it is *incompletable*. No provision of contingent explanatory grounds, finite or infinite, can be complete. There is no all-encompassing

26. See Leibniz's "On the Ultimate Origination of Things," translated in Leibniz 1989, 149–54. Original in Leibniz 1875–90, 7:302–8.

state of explanation—of everything's having been explained—for contingent truth.

This can sound like a pathology, but it need not be. The concept *contingent truth* can be perfectly in order and yet indefinitely extensible, and this fact about it can then be understood as a reflection of the inexhaustibility of PSR's explanatory demand. It may be tempting to reinterpret PSR as expressing only an ideal of inquiry to which explanation always aspires even if, unintelligibly to us, the world does not cooperate. There is something to be said in favor of taking this view, but I think in the present context it concedes too much. For the objection to traditional PSR that lies behind it is mistaken. There is no opposition between the idea that contingent truths must have sufficient reasons and the thought that the concept *contingent truth* is indefinitely extensible. Contingent explanatory grounds may well constitute satisfactory explanations; they just always allow—indeed insist—that the inquiry also be continued, its spade never turned.[27]

The illusion to be dispelled is the idea that satisfactory explanation must consist in some ultimate state in which all possible questions are answered and nothing could be left to ask. Or, rather, that's one of a pair of opposing illusions, the other of which is that explanation runs out at a definite point with brute facts for which there are questions but nothing left to say in answer. Together they pose a false dichotomy of fixity. Honoring the demand for explanation does not require positing a completed state in which all possible questions are answered. Accepting that there is no such state does not require leaving some questions unanswerable.

What's the cost of giving up the illusions of fixity and the idea that there is a completed domain of contingent truth? Sometimes PSR has been interpreted as insisting on an idea of a sufficient reason not only as entailing and explaining the truth for which it is an explanatory ground but also as yielding a complete explanation, one that leaves nothing left to explain. Leibniz himself gives voice to this attitude when he says that even if we might explain the presence of each book in an endless series by appeal to its predecessor, "we still would not have a *complete* explanation" (Leibniz 1989, 149; emphasis in original/Leibniz 1875–90, 7:302). If completeness of explanation—in the strong sense of answering every possible question—is expected of a sufficient reason, that hope will be

27. This comes close to one interpretation of Kant's idea of a "regulative" principle. See Walden, n.d.; and for some related discussion of indefinite extensibility in normative principles, see Walden 2015.

disappointed in the case of contingent truths. But here I think we should reply to Leibniz that a demand for a complete explanation simply overcharacterizes the actual explanatory impulse we express when we articulate PSR. Notice that the clarion defenses of PSR, like Della Rocca's, in fact proceed by showing the power of the demand at any point to press ever further for explanation, not by asking for everything possible all at once. Fixed incompleteness would be a problem; incompletability of the sort being proposed, I submit, captures the explanatory demand as it actually presents itself.

I think this is true of many of PSR's characteristic philosophical uses as well. Appeals to PSR to support "shift" arguments against absolute time or space,[28] or to defend the principle of the identity of indiscernibles,[29] or to argue for the existence of God as a conserving cause,[30] for example, do not require the strong idea that explanations be complete. Even great rationalist versions of the "cosmological" arguments for the existence of God need not always be construed to demand completeness. Positing a necessary being whose action brings about the existence of the infinite series of contingent beings could still allow for details of the necessary being's action to remain contingent and so open to further inquiry. Likewise, I suspect, for the usual "first cause" versions of the argument.

6. Implications of the Loss of Completeness

Giving up the presupposition of Completeness and the idea of "all contingent truths" has substantial implications and raises a number of interesting questions, and the lessons of the foregoing inquiry can indicate some potential answers. Time to gather a few roses for further study.

Some classic philosophical appeals to PSR likely do rely on the idea of a completed domain of all contingent truths. For example, it is not too hard to read Leibniz's (1875–90, 7:302) argument for a metaphysical ground of being that lies outside the entire infinite succession of states of the world (for which the series-of-books example was a stepping-stone) as itself shorthand for a highly general proof that wraps up all contingency in a single embrace and then looks outside of it for a further explanatory ground. If so, his argument must be rejected as imposing a

28. See Leibniz 1875–90, 7:371–72.
29. Ibid.
30. See Descartes 1964–76, 7:165f.

hyperbolic explanatory demand. Indeed, Leibniz's reasoning would then be seen to set the stage for the paradox in the most direct way. Careful historical analysis would be called for to see whether other notable arguments in philosophy might have to be sacrificed under the proposed view of PSR as carrying an incompletable demand for explanation in the case of contingent truths.

A completed domain of all contingent truths has been presupposed in other philosophical quarters, quite apart from PSR, and its denial has intriguing consequences. Classic conceptions of omniscience will need to be retooled, for example. If there is no such thing as all contingent truths, then there is no such thing as knowing all truths. But again, the real specter for omniscience would be incompleteness—truths that are not known—and that's no consequence of the indefinite extensibility of *contingent truth*. Divine knowledge would have to be incompletable, but not incomplete. Defenders of omniscience could reinterpret their position in terms of the denial of counterexamples: there are no truths that are not known. For any truths at all, an omniscient god knows them; there's just no knowing all contingent truths at once.[31]

A perhaps more striking potential implication arises for familiar model-theoretic analyses of modal discourse ("possible-worlds semantics"). Such analyses tend to assume a completed domain of all contingent truths when they treat individual possible worlds as "maximal consistent" sets of propositions or "total" states of affairs. In common apparatuses for describing model theory for modal language, it is held that for every world w and for every proposition or state of affairs P, either w contains P or else w contains $\sim$P. (Worlds are "negation complete.") The actual world is the possible world in which every proposition is true or every state of affairs is actual. If there is no such thing as all contingent truths or all actual contingent states of affairs, though, it seems that there will be no such maximal set or total state of affairs. And so no actual world. Indeed, no possible worlds at all.[32]

What to make of this? Interpretations again multiply. A bullet-biting defender of PSR might now say, "So much the worse for possible-

31. "Open theism" includes a similar view of divine omniscience. It's of interest to see that no claims about the metaphysics of time or free will are required to reach this conclusion.

32. "Modal realism" about possible worlds is no safe harbor either, as nearly the whole train of argument of this essay could be recast replacing 'truth' with 'existence' to show that the concept *contingent existence* is indefinitely extensible.

worlds semantics." On the other hand, one might use that same line of thought as a proof that if there is an actual world, or any possible world *w* at all, it cannot contain contingent truths in the first place. So, given PSR, there cannot be contingent truths—thus a new proof that PSR entails necessitarianism after all. Alternatively, one might take the lesson to be that, given PSR, there is no such thing as possibility or necessity. Or one might even take it to be that, given PSR, there is nothing at all.

I would counsel a more modest interpretation of the "paradox" here, though one in the camp of the bullet-biters. Unreformed model theory has always been an imperfect fit for the analysis of modality, even on its own terms. Entirely aside from the idea of indefinite extensibility, it has already long been in the cards that there can be no such thing as the set of all true propositions or the total actual state of affairs, as these too fall prey to the paradoxes.[33] Russell's paradox of propositions shows this in the most direct way. Replace 'proposition' with 'true proposition' in Russell's argument, and the result is a proof that there cannot be a set of all true propositions—and so no "actual world" for the model theory of modality. Minor adjustments will likewise yield a proof that there are no possible worlds, so understood, at all. So the initial definition of a possible world as a maximal consistent set or a total state of affairs must be reformulated before possible-worlds semantics can even get off the ground.

Dodges are available here for the completist to avoid contradiction. Consider, for example, Harry Deutsch's (2014, 28–30) nice observation that Russell's paradox of propositions can be resolved if—following the example of von Neumann's 'set' versus 'proper class'—we posit a distinction between those propositions that are *members* and those that are *nonmembers*, where the latter cannot be an element of any set.[34] What the paradox can then be said to show is that not all propositions can be members; some must instead simply stand outside of sets. The step in Russell's paradox that forms a set *w* of those characteristic propositions that do not belong to the sets they characterize is thus rejected on the

33. Despite long being in the cards, the point seems to have been late to appear in the literature. The earliest example I can find is Grim 1984, which employs Cantor's Theorem. Menzel (2012) is the first to note how Russell's paradox of propositions yields a disproof of the existence of a maximal consistent set of all true propositions, on some natural assumptions about propositions. See also Uzquiano 2015b.

34. Like Russell's paradox of propositions itself, which can be formulated not only for sets but also for proper classes, Deutsch's distinction is broader: a nonmember is something that cannot be an element of any class, whether 'class' is taken as a set or a proper class. The simpler formulation in terms of sets will serve present purposes.

grounds that those characteristic propositions cannot *all* be members and so there cannot be such a set as *w*. Restricting set formation to only those propositions that are members then yields at most a set w^* that contains all those member characteristic propositions that do not belong to the sets they characterize. And now Russell's paradox can be interpreted to show that w^*'s own characteristic proposition p_{w^*} itself can never be a member. Perhaps with Deutsch's distinction, possible worlds might be redrawn as maximal consistent sets of member propositions, the nonmembers being exiled to some extraworldly semantic stratum—say, true *of* worlds but never true *in* them. Even if this should lead to a workable apparatus, though, it is easy to suspect that the distinction between 'members' and 'non-nonmembers' is another case of wielding the big stick. Why should only *some* propositions be able to belong to any sets?

Other ways of restricting what propositions can belong to worlds—or what propositions there are, or what they can express—may likewise try to elude Russell's paradox of propositions to sustain the completist view. Russell himself (1903, 528) noted that if material equivalence implies identity for propositions, the contradiction is blocked, since the premise that each set of propositions has a *uniquely* corresponding characteristic proposition would fail. The same would be true if necessarily equivalent propositions are identical (Menzel 2012, Uzquiano 2015b). More subtly, Gabriel Uzquiano (2015b) has suggested that propositions might be fine grained enough for necessary equivalence not to imply identity and yet be coarse grained enough not always to be able to distinguish between sets with distinct members, so that the very same characteristic proposition p might speak with a forked tongue to say of one set S, "Every member of S is true," and yet also say of a *different* set S*, "Every member of S* is true." ("Desperate times," remarks Uzquiano [2015b, 341].)

I hope by this point, however, it's clear that embracing a completist tactic for saving the familiar form of model theory for modality from contradiction is not mandatory as an interpretation of the paradox, and not even mandatory as a modest interpretation.[35] In this case too there ought to be an extensibilist alternative that declines to accept the distinctions alleged between 'member' and 'nonmember' or 'true in' and 'true of' (or coarse-grainedness or forked tongues for propositions) but instead looks to reconfigure model theory along different lines. This

35. For a revision of model theory for modality based on "object theory" (and defining possible worlds without sets), see Bueno, Menzel and Zalta 2013.

alternative would not start with the completist's basic premise—common to all the above proposals and the paradox itself—that there is such a thing as "absolutely all propositions." In fact Russell's paradox can be reformulated as a proof that there is no such thing as absolutely all propositions. (More carefully: if propositions are fine grained enough to discriminate between different sets, there can be no such thing as absolutely all propositions.) Let me also note, in passing, that the proof can be converted to show that the concepts *proposition* and *true proposition*, like *contingent truth*, are indefinitely extensible. In for a penny, in for a pound.

The possibility of such an alternative construal of model theory for modality I shall have to leave for now as a conjecture (though I suspect those most likely to want to see such an account can probably craft one for themselves). The point is that we should expect the familiar dialectic to play out again. The objection that indefinite extensibility spoils model theory for modality makes the familiar mistake of presupposing a standard interpretation of the subject at issue in order to criticize a nonstandard rival, when the assumptions of the standard view are themselves in question. It is also worth remembering that the usual model-theoretic approach is just one technique for analyzing modal terms. If it should be insisted that a completed domain of contingent truths is required for sufficiently general possible-worlds semantics, an advocate of indefinite extensibility can turn to other interpretations of modal terms altogether (say, a proof-theoretic semantics). There was no promise that the extensibilist approach to contingent truth would leave everything else just as it was before.

Stepping back a bit further still, it can be seen that a question arises concerning the basic prospects for generality of reasoning about contingent truth. Can anything be held absolutely generally of contingent truths if there is no such thing as all contingent truths? This is not unique to the present case; the same would hold for any given category F once the idea of "absolutely all Fs" is disavowed in favor of an extensibilist conception. It would be particularly ironic in this case if absolute generality were lost, since PSR itself presumably aims to make an absolutely general claim about truth and explanation. But one should not infer too quickly from the disavowal of Completeness to a surrender of absolute generality. I have stated PSR as saying that for any truth there is a sufficient reason. Its corollary for contingent truths in particular would likewise read: for any contingent truth, there is an explanatory ground. By casting those claims in terms of 'any' I have intended to exploit—while simply taking for granted—a plasticity in the ordinary meaning of 'any' to express an absolutely general claim about contingent truths without thereby assuming that there is such a thing as absolutely all contingent truths. (I mean the sense of ‘any’ familiar in classroom problem solving in phrases like ‘let *x* be any integer’ or ‘take any *x* and *y*’.) It is controversial how best to make sense of such generality, having set aside key elements of the familiar extensional model of quantification over a completed domain. There are various options to try for an extensibilist, perhaps taking a page from the classic intuitionistic interpretation of the quantifiers, for example, or from Fine’s (2006) “postulational” interpretation, or Parsons’s (1974) “systematic ambiguity,” and so on. It remains to be seen which approach, if any, would be particularly suitable for an extensibilist defender of PSR—more work left to another day—but one need not automatically assume that absolute generality is lost.

A final thought is in order, one that encompasses the same genetic materials carried in the previous concerns. Rejecting Completeness in favor of indefinite extensibility asks us to abandon a natural way of thinking about the world—namely, as a single all-inclusive domain open to comprehensive inquiry into its elements. Just which among the elements of reality are the contingent truths? The natural picture invites us to expect that if the question truly has an answer, we could “see them all at once” if only we could occupy the right standpoint. But if the concept *contingent truth* is indefinitely extensible, there can be no such standpoint. Reality is not surveyable in this way. An answer to the question “Just which things are the contingent truths?” may of course be given, in principle at least. The contingent truths can be separated from the rest exactly through the conditions imposed by the concept. But not all at once. If a simple picture, or model, is wanted, some analogy to illustrate the kind of answer the question receives, it would be an iterative one of discovery by stages.

This does not have to mean that what is discovered is a “mental construction” or something that comes into being via inquiry. An alternative would be to imagine an array of limited perspectives from which to see the world, some more inclusive than others, but none capable of taking in everything that could ever be seen.[36] (“Everything that could ever be seen?” None maximally inclusive: for any perspective, there is a

36. Or a view of “the world” as an iterated sequence of domains, each more inclusive than its predecessors; Zermelo (1930) proposes this about the universe of sets. Either way, however, the extensibilist should regard ‘perspective’ and ‘domain’ here as expressing indefinitely extensible concepts.

more inclusive.) As a parallel here, we might turn to Gödel's (1995a [1933], 1995b [1951]) view of set theory as incompletable despite its being a theory of a mind-independent mathematical world. However expansive our body of axioms, we will always, by reference to what we have codified, be able to identify further sets or truths that lie beyond the reach of those axioms but yet clearly satisfy the concepts involved. Even the faculty of "mathematical perception" that Gödel believed to be turned upon an objective mathematical landscape was not presented as an all-seeing eye. Likewise now for the landscape of contingent truth.

It is hardly a novel suggestion to urge relinquishing the idea of a God's-eye point of view and its once-for-all survey of the possible and the actual sub specie aeternitatis. If rationalism is the idea that everything is explicable, maybe it is time to take seriously that explicability is something that allows only a progressive or piecemeal approach. In that case, there should be nothing unwelcome to a rationalist in acquiring a tincture of constructivism.

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